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# Thermal Fermionic Dispersion Relations in a Magnetic Field

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**Abstract**

The thermal self-energy of an electron in a static uniform magnetic field $B$ is calculated to first order in the fine structure constant $\alpha$ and to all orders in $eB$. We use two methods, one based on the Furry picture and another based on Schwinger's proper-time method. As external states we consider relativistic Landau levels with special emphasis on the lowest Landau level. In the high-temperature limit we derive self-consistent dispersion relations for particle and hole excitations, showing the chiral asymmetry caused by the external field. For weak fields, earlier results on the ground-state energy and the anomalous magnetic moment are discussed and compared with the present analysis. In the strong-field limit the appearance of a field-independent imaginary part of the self-energy, related to Landau damping in the $e^+e^-$ plasma, is pointed out.

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# 1 Introduction

It has become increasingly important to understand the quantum field theory of elementary processes in the presence of both a thermal heat bath and strong background fields, in particular in connection with applications to astrophysical and cosmological models. A natural strategy for making progress in this area is to generalize existing background field calculations to finite temperature and vice versa. The first obvious quantity to study is the effective potential, which governs the thermodynamics of the system, and there are several papers on this topic [1–7]. Next, we are interested in more detailed kinematical issues, which are determined by the propagation of the elementary excitations in a plasma. Dispersion relations for electrons have been studied in great detail in strong background fields but without any heat bath [8–14], and at high temperature but in the absence of external fields [15–19].

In this paper we perform a detailed study of the fermion self-energy in QED, taking into account both the effects from a magnetic background field and a thermal heat bath. In particular we consider the limits of strong fields in the lowest Landau level and weak fields at high temperature. There are two basic methods for doing perturbative quantum field theory calculations in a constant magnetic background field. One is the Furry picture, where the fermion propagator is constructed by an explicit sum over the solutions to the Dirac equation [20]. The other one is the Schwinger proper-time method, where the propagator is expressed directly in terms of operators without any reference to an explicit representation of the states [21]. The different methods are suitable for different calculations and we have used them both. Typically, the Furry picture is convenient in the strong field limit where only a few Landau levels contribute, while Schwinger's method is particularly useful for weak fields.

We have organized the paper as follows. The two basic methods, the Furry picture and Schwinger's proper-time method, are introduced in Sections 2 and 3. The weak field limit is studied in Section 4 using both methods. It is particularly interesting to find the dispersion relation at high temperature for weak fields, as we do in Section 5, since it is then consistent to not only calculate first-order corrections but to solve the dispersion relation self-consistently. As an application of the weak field expansion we study the anomalous magnetic moment in Section 6 and compare it with other calculations. The strong field limit, where only the lowest Landau level contributes, is investigated in Section

8. Discussions of the results follow in Section 9, and the appendices contain some technical details.

## 2 Self-energy corrections of the lowest Landau level in the Furry picture

We consider Dirac fermions with charge $-e$ in the presence of an external field described by the vector potential $A_\mu = (0, 0, Bx, 0)$, corresponding to a static uniform magnetic field in the negative $z$-direction. Using static energy solutions to the Dirac equation $(i\partial\!\!\!/ + eA\!\!\!/ - m)\Psi_\kappa^{(\pm)} = 0$, we may represent the second quantized fermion field in the Furry picture [20] as

$$\Psi = \sum_\kappa \left[ b_\kappa \Psi_\kappa^{(+)}(\mathbf{x}, t) + d_\kappa^\dagger \Psi_\kappa^{(-)}(\mathbf{x}, t) \right] \ , \tag{2.1}$$

where $\kappa$ denotes a complete set of quantum numbers and $(b, d)$ are the standard annihilation operators for particles and antiparticles [22]. The fermion propagator, including the effects of some distribution of particles, can then be constructed explicitly as the expectation value (see [4] for more details)

$$iS(x', x) = \langle \mathbf{T} \left[ \mathbf{\Psi}(\mathbf{x}')\overline{\mathbf{\Psi}}(\mathbf{x}) \right] \rangle \quad . \tag{2.2}$$

Evaluating the expectation values of the creation and annihilation operators we may separate into vacuum and thermal (actually more generally due to some arbitrary distribution of fermions) contributions, $S(x', x) = S_{\text{vac}}(x', x) + S^{\beta,\mu}(x', x)$. We find

$$iS_{\text{vac}}(x', x) \ = \ \sum_\kappa \left[ \Theta(t' - t)\Psi_\kappa^{(+)}(x')\overline{\Psi}_\kappa^{(+)}(x) - \Theta(t - t')\Psi_\kappa^{(-)}(x')\overline{\Psi}_\kappa^{(-)}(x) \right] \quad , \tag{2.3}$$

$$iS^{\beta,\mu}(x', x) \ = \ \sum_\kappa \left[ f_F^+(E_\kappa)\, \Psi_\kappa^{(+)}(x')\overline{\Psi}_\kappa^{(+)}(x) - f_F^-(E_\kappa)\Psi_\kappa^{(-)}(x')\overline{\Psi}_\kappa^{(-)}(x) \right] \quad , \tag{2.4}$$

where $E_\kappa$ is the energy eigenvalue of $\Psi_\kappa^{(\pm)}$. For fermions, the antiparticle distribution is determined by $f_F^-(k_0) \equiv 1 - f_F^+(-k_0)$. We shall here consider only the case of thermal equilibrium:

$$f_F^\pm(k_0) = \frac{1}{e^{\beta(k_0 \mp \mu)} + 1} \ , \tag{2.5}$$

where $\beta$ is the inverse temperature and $\mu$ is the chemical potential determined by the charge density of electrons and positrons of the system. For the explicit form of the wave-functions and of the propagator we have summarized the relevant expressions in Appendix A.

To lowest order in the electromagnetic coupling there are two possible contributions to the electron self-energy, the one-particle irreducible self-energy diagram and the tadpole. In Appendix C we show that the tadpole gives no contribution to the self-energy in a neutral background. Therefore, in configuration space the self-energy is to one-loop order given by

$$- i\Sigma(x', x) = (-ie)^2 \gamma_\mu i D^{\mu\nu}(x' - x) i S(x', x)\gamma_\nu \quad . \tag{2.6}$$

In a general covariant gauge, parametrized by $\xi$, the photon propagator is given by [23–25]

$$iD^{\mu\nu}(x) = \int [d^4q] e^{-iq\cdot x} \left( g^{\mu\nu} - \xi\, q^\mu q^\nu \frac{\partial}{\partial q^2} \right) \left[ \frac{-i}{q^2 + i\varepsilon} - 2\pi\delta(q^2) f_B(q_0) \right] \quad , \tag{2.7}$$

where the photon distribution function in the case of thermal equilibrium is

$$f_B(q_0) = \frac{1}{\exp[\beta|q_0|] - 1} \quad , \tag{2.8}$$

and the operator $\partial/\partial q^2$ is not supposed to act on the $q_0$ in $f_B(q_0)$. We have written $[d^n q] \equiv d^n q/(2\pi)^n$ as a short-hand notation for the integration measure. The expectation value, $\Delta E$, of the self-energy in a state described by the wave function $\Psi^{(+)}_{\zeta;n,p_y,p_z}(x)$ is defined by the expression

$$\Delta E_{\zeta;n,p_y}(p_z) = \frac{\int d^4x\, d^4x'\, \overline{\Psi}^{(+)}_{\zeta;n,p_y,p_z}(x')\Sigma(x', x)\Psi^{(+)}_{\zeta;n,p_y,p_z}(x)}{\int dy\, dz\, dt} \quad , \tag{2.9}$$

where $\int dy\, dz\, dt$ is a normalization factor from the continuous spectrum in $p_y$, $p_z$, since $\int d^4x \Psi^\dagger_\kappa(x)\Psi_\kappa(x) = \int dy\, dz\, dt$, cf. Eq. (A.16). In the gauge we use, the translational invariance in the plane perpendicular to the $B$ field is reflected by the energy degeneracy for different values of $p_y$. Explicit calculations show indeed that $\Delta E_{\zeta;n,p_y}(p_z)$ is independent of $p_y$ in the lowest Landau level, so we consider for simplicity only $p_y = 0$ here. We formally show in Appendix D that the self-energy is independent of the gauge-fixing parameter $\xi$ on the tree-level mass shell. This is explicitly verified in the high-field limit in Section 8.2. We shall therefore use the Feynman gauge $\xi = 0$ for the photon propagator. From the inverse propagator to one-loop order, the effective Dirac equation is obtained as

$$(i \not{D} - m)\Psi(\mathbf{x}, t) = \int d^4x'\, \Sigma(x, x')\Psi(x') \quad , \tag{2.10}$$

where $D_\mu = \partial_\mu - ieA_\mu$. Let us consider the value of the energy $E$, appearing in the wave functions for particles in Eq. (A.3) as $\exp[iEt]$, as a free parameter not to be constrained

by the tree-level Dirac equation. Then the expectation value of Eq. (2.10) leads to the dispersion relation

$$E = E_n(p_z) + \Delta E(p_z) \quad , \tag{2.11}$$

where $E_n(p_z) = \sqrt{m^2 + p_z^2 + 2eBn}$, and $\Delta E$ is to be calculated on the tree-level mass shell. It is tempting to try to solve Eq. (2.11) self-consistently in $E$, by substituting $E_n \to E$ in the phase factor in $\Psi$ before taking the expectation value, and calculate $\Delta E$ with such an arbitrary $E$. However, this is inconsistent for several reasons. First, not only $E$ but the actual form of the wave functions should then also be solved self-consistently. Secondly, it turns out that the self-energy is in general only gauge-invariant on the tree-level mass shell and a self-consistent solution would become gauge-dependent. In [26] a fermion propagator equivalent to the one used here, but written explicitly in terms of $\gamma$ matrices, was derived. Using this propagator and the general ansatz for the wave functions in Eqs. (A.3) and (A.10), it is possible to calculate the general matrix structure of the self-energy operator, that acts on the space-independent spinor $u_\kappa$, in the Furry picture. Then one could proceed as in the following sections, where instead a generalization of Schwinger's proper-time method has been employed, and solve the self-consistent dispersion relation in a limit, such as the high-temperature limit, where the off-shell gauge dependence may be neglected.

Here we shall confine ourselves to the calculation of the energy shift for an electron in the lowest relativistic Landau level. Suppressing the $\zeta = 1,\ n = 0,\ p_y = 0$ subscripts and the $B$ and $p_z$ dependence, we separate the self-energy into its contributions from the vacuum, thermal fermions, and thermal photons

$$\Delta E = \Delta E_{\text{vac}} + \Delta E^{\beta;\mu}_{e^+e^-} + \Delta E^{\beta}_{\gamma} \quad , \tag{2.12}$$

respectively. In [12] the vacuum part was calculated for $p_z = 0$, using the methods adopted here. Introducing an arbitrary momentum parallel to the external field does not alter anything in the vacuum sector, since the relativistic invariance is unbroken in the $z$ and $t$ directions. The self-energy is then a function of only $E_0^2 - p_z^2 = m^2$ and $eB$. We have confirmed this by explicit calculations, of which we only quote the result here,

$$\begin{aligned}\Delta E_{\text{vac}} &= \frac{m^2}{E_0}\frac{\alpha}{2\pi}\int_0^1 ds \int_0^\infty du\, \exp(-m^2 u s^2) \\ &\times \left\{ \frac{2eB\left[1 + s\, e^{-2eBus}\right]}{1 + 2eBu(1-s) - e^{-2eBus}} - \frac{1+s}{u} \right\} \quad .\end{aligned} \tag{2.13}$$

The second term on the right-hand side comes from the ordinary mass renormalization to make the vacuum part of the self-energy vanish for $B = 0$.

Using the thermal electron propagator given in Appendix A, we have

$$
\begin{aligned}
\Delta E^{\beta;\mu}_{e^+e^-} &= -i\frac{2e^2}{E_0}\frac{\int d^4x\, d^4x'}{\int dy'\, dz'\, dt'} \exp[iE_0(t-t') - ip_z(z'-z)] I_{0;0}(x) I_{0;0}(x') \\
&\quad \times \int [d^4q] \frac{\exp[-iq\cdot(x-x')]}{q^2+i\varepsilon} \\
&\quad \times \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} [dk_0]\, [dk_y]\, [dk_z] \exp[-ik_0(t'-t) + ik_y(y'-y) + ik_z(z'-z)] \\
&\quad \times 2\pi i \delta(k_0^2 - m^2 - k_z^2 - 2eBn)\, f_F(k_0) \\
&\quad \times \Big\{ m^2 I_{n;k_y}(x) I_{n;k_y}(x') + \Big[m^2 - E_0k_0 + p_zk_z\Big] I_{n-1;k_y}(x) I_{n-1;k_y}(x') \Big\} \quad .
\end{aligned}
\tag{2.14}
$$

The $t, y$ and $z$ integrals are trivially performed, and produce $\delta$-functions that are used to perform the corresponding $q$ integrals. We now express $I_{n;k_y}$ in terms of the explicit form in Eq. (A.5), and use the fact that $H_0(x) = 1$. Then the variables $x, x', k_x \equiv q_x$ and $k_y$ are shifted and rescaled. Fourier-transforming the $\delta$-function

$$
\delta(k_0^2 - m^2 - k_z^2 - 2eBn) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds \exp[-is(k_0^2 - m^2 - k_z^2 - 2eBn)] \quad , \tag{2.15}
$$

and using the Feynman prescription to write

$$
\frac{1}{a+i\epsilon} = -i\int_0^{\infty} du\, e^{iu(a+i\epsilon)} \quad , \tag{2.16}
$$

with $a = (k_0 - E_0)^2 - (k_z - p_z)^2 - 2eB(k_x^2 + k_y^2) + i\varepsilon$, we find

$$
\begin{aligned}
\Delta E^{\beta;\mu}_{e^+e^-} &= -i\frac{2e^2}{(2\pi)^5}\frac{2eB}{E_0}\int_{-\infty}^{\infty} dk_0\, f_F(k_0) \int_{-\infty}^{\infty} ds \int_0^{\infty} du \int dx\, dx'\, dk_x\, dk_y\, dk_z \\
&\quad \times \exp\Big\{-is[k_0^2 - m^2 - k_z^2] + iu[(k_0-E_0)^2 - (k_z-p_z)^2 - 2eB(k_x^2 + k_y)^2 + i\varepsilon]\Big\} \\
&\quad \times \exp\Big[-\frac{1}{2}(x^2 + x'^2) - k_y(x+x') + ik_x(x-x') - 2k_y^2\Big] \\
&\quad \times \Big[m^2 + (m^2 - E_0k_0 + p_zk_z)\lambda\Big] \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x) H_n(x') \quad ,
\end{aligned}
\tag{2.17}
$$

where we have shifted the summation in the $I_{n-1,n-1}$ term, and defined $\lambda \equiv e^{i2eBs}$. Let us now use the integral representation of Hermite polynomials [27]:

$$
\exp[-x^2/2]\, H_n(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dv\, \exp[-v^2/2 + ivx](-iv)^n \quad , \tag{2.18}
$$

and similarly for $H_n(x')$ in terms of an integral over $v'$. We may then identify an exponential function

$$\sum_{n=0}^{\infty} \frac{\lambda^n(-iv)^n(-iv')^n}{n!} = \exp[-vv'\lambda] \quad . \tag{2.19}$$

We now have some Gaussian integrals that may be performed in the order $v, x, v'$, and finally $x'$. Let us introduce a factor of convergence $\exp[-2eB\,\epsilon\,(k_x^2 + k_y^2)]$, where $\epsilon \to 0^+$. We may now also change the order and perform the $k_x, k_y$ integrals, while keeping the $s, u$ parameter integrals. The final result then reads

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-} \;=\; & -i\frac{\alpha}{\pi^2}\frac{eB}{E_0}\int_{-\infty}^{\infty} dk_0\, f_F(k_0) \int_{-\infty}^{\infty} ds \int_0^{\infty} du \int dk_z \exp\{-is[k_0^2 - m^2 - k_z^2]\} \\ & \times \exp\{iu[(k_0 - E_0)^2 - (k_z - p_z)^2 + i\varepsilon]\} \frac{m^2 + (m^2 - E_0 k_0 + p_z k_z)e^{i2eBs}}{1 + 2eB(\epsilon + iu) - e^{i2eBs}} \quad . \end{aligned} \tag{2.20}$$

The Gaussian integral over $k_z$ could easily be performed, but we prefer to keep it in order to make the $s, u$ integrals simpler when considering the weak field limit.

The thermal photon contribution is obtained in a way analogous to the case of thermal electrons. The final result reads

$$\begin{aligned}\Delta E^{\beta}_{\gamma} \;=\; & i\frac{\alpha}{\pi^2}\frac{eB}{E_0}\int_{-\infty}^{\infty} dk_0\, f_B(k_0) \int_{-\infty}^{\infty} ds \int_0^{\infty} du \int dk_z \exp\{-is[k_0^2 - k_z^2]\} \\ & \times \exp\{iu[(k_0 + E_0)^2 - m^2 - (k_z + p_z)^2 + i\varepsilon]\} \frac{m^2 - (E_0 k_0 - p_z k_z)e^{-i2eBu}}{1 + 2eB(\epsilon - is) - e^{-i2eBu}} \quad . \end{aligned} \tag{2.21}$$

The two equations (2.20) and (2.21) serve as starting points for the weak field expansion in Section 4.

# 3 Schwinger's proper-time method

In 1951 Schwinger [21] used the analogy with a quantum mechanical problem to find an explicit expression for the electron propagator in an external electromagnetic field without first finding the solutions to the Dirac equation in the field. Later, mostly in the 70's, this method was used to calculate a number of different properties of electrons in external fields. The real-time finite-temperature propagator can be constructed in a simple way from the zero-temperature propagator. Therefore, we shall in this section use Schwinger's method to calculate the electron self-energy in a magnetic field at finite temperature, in order to

compare with the corresponding calculations using the Furry picture, and to generalize to the full self-energy operator. We use standard real-time rules and as long as we are only interested in the real part of the self-energy to one-loop it is sufficient to calculate the (11)-part of the propagator. The fact that standard Thermo Field Dynamics works also in a magnetic field should be obvious, and it was discussed in [4, 5]. (For the imaginary part we can use the rules in [28].) The propagator then is

$$iS(x',x'') = \langle x'|\frac{i}{\not{\Pi} - m + i\varepsilon} - f_F(p_0)\left(\frac{i}{\not{\Pi} - m + i\varepsilon} - \frac{i}{\not{\Pi} - m - i\varepsilon}\right)|x''\rangle \ , \tag{3.22}$$

where $f_F(p_0)$ is defined in Eq. (A.18) and $\not{\Pi} = \gamma^\mu(p_\mu + eA_\mu)$,as usual. We shall not repeat Schwinger's calculation, but just give the result for the zero temperature part in our notation

$$\begin{aligned}
\langle x'|\frac{i}{\not{\Pi} - m + i\varepsilon}|x''\rangle &= \frac{-i}{(4\pi)^2}\phi(x',x'')\int_0^\infty \frac{ds}{s}e^{-L(s)}\exp\left[-is\left(-\frac{e\sigma F}{2} + m^2 - i\varepsilon\right)\right] \\
&\quad \times \exp[-\frac{i}{4}(x'-x'')eF\coth eFs(x'-x'')] \\
&\quad \times \left\{\frac{\gamma}{2}(eF\coth eFs + eF)(x'-x'') + m\right\} \\
&\equiv \phi(x',x'')\int[d^4p]e^{-ip(x'-x'')}iS_{\mathrm{vac}}(p) \ ,
\end{aligned} \tag{3.23}$$

where we have suppressed Lorentz indices ($F = F_\mu{}^\nu$) and used the notation

$$\begin{aligned}
\phi(x',x'') &= \exp\left[ie\int_{x''}^{x'} dx^\mu\left(A_\mu + \frac{1}{2}F_{\mu\nu}(x-x'')^\nu\right)\right] \ , \\
\exp[-L(s)] &= \frac{e^2s^2ab}{\sin(eas)\sinh(ebs)} \ , \\
a^2 - b^2 &= B^2 - E^2 \ , \\
ab &= \mathbf{B}\cdot\mathbf{E} \ .
\end{aligned} \tag{3.24}$$

The phase factor $\phi(x',x'')$ is the only part of Eq. (3.23) that depends explicitly on the gauge, and that part factors out in a natural way when we compute the self-energy. Restricting the field to be purely magnetic in the negative $z$-direction we find in momentum space ($\sigma_z \equiv \sigma_{xy}$)

$$\begin{aligned}
iS_{\mathrm{vac}}(p) &= \int_0^\infty ds\frac{e^{ieBs\sigma_z}}{\cos eBs}\exp\left[is\left(p_\parallel^2 - \frac{\tan eBs}{eBs}p_\perp^2 - m^2 + i\varepsilon\right)\right] \\
&\quad \times\left\{\gamma p_\parallel - \frac{e^{-ieBs\sigma_z}}{\cos eBs}\gamma p_\perp + m\right\} \ ,
\end{aligned} \tag{3.25}$$

where for general four-vectors $a$ and $b$, $a \cdot b_{\|} = a_0 b_0 - a_z b_z$ and $a \cdot b_{\perp} = a_x b_x + a_y b_y$. In order to calculate the thermal part we choose a gauge where $A_0 = 0$ and $\partial_0 A_\mu = 0$. This is very natural for a static magnetic field (for a further discussion of the gauge dependence of the thermal propagator in a background field, see [5]). The $\phi(x', x'')$ can then also be factorized out in front of the thermal propagator. At finite temperature we see from Eq. (3.22) that Eq. (3.25) should be replaced by

$$iS_{\rm vac}(p) - f_F(p_0)\Big(iS_{\rm vac}(p) - iS^*_{\rm vac}(p)\Big) \ . \tag{3.26}$$

The real combination that occurs in the thermal part of Eq. (3.26) is finally obtained by extending the $s$-integral in Eq. (3.25) from $-\infty$ to $\infty$. In the integrand of Eq. (3.25) there are poles and essential singularities on the real $s$ axis. They have to be avoided by taking the integration contour in the lower half-plane for positive $s$ (see e.g. [4] for a discussion of this contour); to get a real quantity for the thermal part it has, therefore, to go in the lower half-plane also for negative $s$. These poles are similar to the ones responsible for the non-perturbative pair production in an external electric field [21] or the de Haas–van Alphen oscillations in a degenerate electron gas [4]. When we consider the weak field limit in Section 4 we do not encounter these poles to leading order.

The self-energy may be represented in momentum space as

$$-i\Sigma(x', x'') = -i\phi(x', x'') \int [d^4p] e^{-ip(x'-x'')} \Sigma(p) \ , \tag{3.27}$$

where the only breaking of translational invariance is in the phase factor $\phi(x', x'')$. There are two contributions to the real part of the thermal self-energy at one loop. One from thermal photons and one from thermal electrons. Let us start with the photon contribution. The electron contribution is completely analogous. The thermal part of the photon propagator may be represented in the Feynman gauge by

$$D^{\mu\nu}_{\beta}(k) = -g^{\mu\nu} f_B(k_0) \left( \frac{i}{k^2 + i\varepsilon} - \frac{i}{k^2 - i\varepsilon} \right) = -g^{\mu\nu} f_B(k_0) \int_{-\infty}^{\infty} dt \exp[itk^2 - |t|\,\varepsilon] \ , \tag{3.28}$$

and the corresponding contribution to the self-energy is

$$
\begin{aligned}
\Sigma_\gamma^\beta(p) \;=\;& ie^2\gamma^\mu \int [d^4k] f_B(k_0) \int_{-\infty}^{\infty} dt \exp[itk^2 - |t|\,\varepsilon] \\
& \times \int_0^\infty ds \exp\left[is\left((p-k)_\parallel^2 - \frac{\tan eBs}{eBs}(p-k)_\perp^2 - m^2 + i\varepsilon\right)\right] \\
& \times \frac{e^{ieBs\sigma_z}}{\cos eBs}\left\{\gamma(p-k)_\parallel - \frac{e^{-ieBs\sigma_z}}{\cos eBs}\gamma(p-k)_\perp + m\right\}\gamma_\mu \\
=\;& \frac{ie^2}{(2\pi)^3}\int_{-\infty}^{\infty}[dk_0] f_B(k_0)dt\,ds\left(\frac{\pi}{i(s+t)}\right)^{1/2}\frac{eB\pi}{i(eBt+\tan eBs)} \\
& \times \exp\left[i\left(tk_0^2 + s(p_0-k_0)^2 - \frac{st}{s+t}p_z^2 - \frac{t\tan eBs}{eBt+\tan eBs}p_\perp^2 - m^2 s\right) - s\varepsilon - |t|\,\varepsilon\right] \\
& \times \gamma^\mu \frac{e^{ieBs\sigma_z}}{\cos eBs}\left\{\gamma_0(p_0-k_0) - \frac{t}{s+t}\gamma_z p_z - \frac{e^{-ieBs\sigma_z}}{\cos eBs}\frac{eBt}{eBt+\tan eBs}\gamma p_\perp + m\right\}\gamma_\mu \ ,
\end{aligned}
\tag{3.29}
$$

where we have followed [9] very closely, the only essential difference being that the $k_0$-integral cannot be carried out explicitly. The $\Sigma_\gamma^\beta$ above is a not an operator but a complicated function of the parameter $p_\mu$. In order to obtain the expectation value of the energy shift in a given state one would have to multiply it with explicit wave functions and integrate over $p_\mu$. There is, however, a clever way to rewrite it as an operator [9] (thus replacing $p_\mu$ with gauge-invariant operators $\Pi_\mu$) which, when acting on the tree level wave functions, has simple properties. Noticing that $\phi(x', x'')$ only depends on $x'_\perp$ and $x''_\perp$, we write

$$
\langle x'|\hat{\Sigma}|x''\rangle = \int [dp_\parallel] e^{-ip(x'-x'')_\parallel}\phi(x',x'') \int [dp_\perp] e^{ip(x'-x'')_\perp}\Sigma(p_\parallel, p_\perp) \ . \tag{3.30}
$$

The key relations to be used are then

$$
\begin{aligned}
& \phi(x',x'') \int [dp_\perp] e^{ip(x'-x'')_\perp} \exp\left[-i\frac{\tan v}{eB}p_\perp^2\right] \\
& \qquad = \langle x'|\exp\left[-i\frac{v}{eB}\Pi_\perp^2\right]|x''\rangle \cos v \ , \\
& \phi(x',x'') \int [dp_\perp] e^{ip(x'-x'')_\perp} \exp\left[-i\frac{\tan v}{eB}p_\perp^2\right]\gamma p_\perp \\
& \qquad = \langle x'|\exp\left[-i\frac{v}{eB}(\Pi_\perp^2 - eB\sigma_z)\right]\not{\Pi}_\perp|x''\rangle \cos^2 v \ .
\end{aligned}
\tag{3.31}
$$

After performing the $\gamma$-matrix algebra we obtain the final expression for the self-energy

operator

$$
\begin{aligned}
\hat{\Sigma}^{\beta}_{\gamma} &= \frac{e^2}{8\pi^2}\int_{-\infty}^{\infty}[dk_0]\int_{-\infty}^{\infty}dt\int_{0}^{\infty}ds\left(\frac{\pi}{i(s+t)}\right)^{1/2}\frac{eB}{eBt+\tan eBs}f_B(k_0) \\
&\times\exp\left[i\left(tk_0^2+s(p_0-k_0)^2-\frac{st}{s+t}p_z^2-\frac{v}{eB}\Pi_\perp^2-m^2s\right)\right] \\
&\times\frac{\cos v}{\cos eBs}\left\{-2e^{-ieBs\sigma_z}\left(\gamma_0(p_0-k_0)-\frac{t}{s+t}p_z\gamma_z\right)+4m\cos eBs\right. \\
&\left.+\frac{2\cos v}{\cos eBs}\frac{eBt}{eBt+\tan eBs}e^{iv\sigma_z}\not{\Pi}_\perp\right\} , \qquad (3.32)
\end{aligned}
$$

where

$$
\tan v=\frac{eBt\tan eBs}{eBt+\tan eBs} . \qquad (3.33)
$$

We notice that $\Pi_\perp^2$ and $\not{\Pi}_\perp$ do not commute but $\not{\Pi}_\perp$ is actually multiplied by a function of only $\Pi_\perp^2-eB\sigma_z$ with which it does commute, so the ordering is not a problem. A similar analysis for thermal electrons gives

$$
\begin{aligned}
\hat{\Sigma}^{\beta}_{e^+e^-} &= -\frac{e^2}{8\pi^2}\int_{-\infty}^{\infty}[dk_0]\int_{0}^{\infty}dt\int_{-\infty}^{\infty}ds\left(\frac{\pi}{i(s+t)}\right)^{1/2}\frac{eB}{eBt+\tan eBs}f_F(k_0) \\
&\times\exp\left[i\left(t(p_0-k_0)^2+sk_0^2-\frac{st}{s+t}p_z^2-\frac{v}{eB}\Pi_\perp^2-m^2s\right)\right] \\
&\times\frac{\cos v}{\cos eBs}\left\{-2e^{-ieBs\sigma_z}\left(\gamma_0k_0-\frac{t}{s+t}p_z\gamma_z\right)+4m\cos eBs\right. \\
&\left.+\frac{2\cos v}{\cos eBs}\frac{eBt}{eBt+\tan eBs}e^{iv\sigma_z}\not{\Pi}_\perp\right\} . \qquad (3.34)
\end{aligned}
$$

The matrix elements of $\hat{\Sigma}^{\beta}_{e^+e^-}$ and $\hat{\Sigma}^{\beta}_{\gamma}$ depend of course on the basis in which they are calculated. A suitable basis that diagonalizes the eigenvalues $\kappa\equiv(E,n,p_y,p_z)$ should, with the chiral representation of $\gamma_\mu$, have the form (see e.g. Appendix A):

$$
\begin{aligned}
\Psi_\kappa(x) &= \exp[-i(Et-p_zz-p_yy)]V_{n,p_y}(x)u(E,n,p_y,p_z) , \\
V_{n,p_y}(x) &= \mathrm{diag}\Big(I_{n,p_y}(x),I_{n-1,p_y}(x),I_{n,p_y}(x),I_{n-1,p_y}(x)\Big) , \qquad (3.35)
\end{aligned}
$$

where $I_{n,p_y}(x)$ is defined in Appendix A and $u(E,n,p_y,p_z)$ is a Dirac spinor, independent of $x_\mu$. With this choice of wave function, the space-time integrals of (see Eq. (2.9))

$$
\langle\Psi_\kappa|\hat{\Sigma}|\Psi_{\kappa'}\rangle=\frac{\int d^4x\,d^4x'\overline{\Psi}_\kappa(x)\Sigma(x,x')\Psi_{\kappa'}(x')}{\int\,dy\,dz\,dt} \qquad (3.36)
$$

can be carried out directly and give a $\delta$-function in $\kappa-\kappa'$. There then remains a $4\times 4$ matrix that can be diagonalized with a suitable choice of the spinors $u(E,n,p_y,p_z)$.

# 4 Weak-field expansion

In many physical applications the magnetic field is strong enough to be important, but still weak enough for an expansion in $eB/m^2$ to be useful. It should, however, be noted that there are some fundamental difficulties with the limit of weak fields. At any finite field strength the eigenfunctions are Landau levels and there is not a single eigenfunctions that continuously goes over to a plane wave when $B \to 0$. The Fourier coefficients of the function $I_{n,p_y}(x)$ in Eq. (A.5) are given by

$$\int dx\, e^{ikx} I_{n,p_y}(x) = i^n \sqrt{\frac{2}{n!}} \left(\frac{\pi}{eB}\right)^{1/4} \exp\left[-\frac{(k-ip_y)^2}{2eB} - \frac{p_y^2}{2eB}\right] H_n\left(k\sqrt{\frac{2}{eB}}\right) \quad , \tag{4.37}$$

and they cannot be expanded in powers of $B$. It is, therefore, not clear that there exists a continuous limit as $B \to 0$ in general. In fact, we know that in a background of degenerate electrons there are de Haas–van Alphen oscillations that do not have a series expansion in $B$ [4]. In standard first-order perturbation theory it is enough to calculate the expectation value of the perturbation in the unperturbed states. It would therefore be tempting to use plane waves as external states for weak fields. But, as explained above, the exact states are not approximately equal to plane waves and we have to use the Landau levels as a basis even for weak fields.

## 4.1 The Furry picture

In order to obtain the weak-field limit we expand the denominator in Eq. (2.20):

$$\frac{-i2eB}{1+2eB(\epsilon+iu)-e^{i2eBs}} = \frac{1}{s-u+i\epsilon} - i\frac{eBs^2}{(s-u+i\epsilon)^2} + \mathcal{O}(eB)^2 \quad . \tag{4.38}$$

In the limit $B \to 0$ we may close the contour and use Cauchy's theorem to integrate over $s$. The simple pole from the first term in Eq. (4.38) gives a contribution for $k_0^2 - m^2 - k_z^2 > 0$. We may then perform also the $t$ and $k_z$ integrals, with the final result

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-}(B=0) &= \frac{\alpha}{2\pi}\frac{m}{E_0}\int_{-\infty}^{\infty} dk_0 \Theta(k_0^2-m^2) f_F(k_0) \\ &\times \left[2\frac{\sqrt{k_0^2-m^2}}{m} - \frac{m}{p_z}\ln\left(\frac{E_0k_0 - m^2 + p_z\sqrt{k_0^2-m^2}}{E_0k_0 - m^2 - p_z\sqrt{k_0^2-m^2}}\right)\right] \quad .\end{aligned} \tag{4.39}$$

This result is finite, well-behaved as $p_z \to 0$, and it agrees with the thermal self-energy calculated in a conventional plane-wave basis. To linear order in $eB$ we use Eq. (4.38) together with $e^{i2eBs} = 1 + i2eBs + \mathcal{O}(eB)^2$ in Eq. (2.20). In order not to get a contribution when closing the $s$ contour by a semi-circle at infinity, it is necessary to rewrite terms such as e.g. $s/(s-u+i\varepsilon) = 1 + u/(s-u+i\varepsilon)$ and treat the 1 separately. The term linear in $eB$ is thus obtained from

$$-\,i\,eB\left[(E_0k_0 - p_zk_z) + 2m^2\frac{u}{s-u+i\epsilon} + (2m^2 - E_0k_0 + p_zk_z)\frac{u^2}{(s-u+i\epsilon)^2}\right] \quad . \qquad (4.40)$$

Integrating over $s$ the first term in Eq. (4.40) gives a contribution proportional to $\delta(k_0^2 - m^2 - k_z^2)$. For the last two terms we proceed as before, close the $s$-contour and then use Cauchy's theorem. Performing the standard integrals over $t$ and $k_z$ we arrive at the term linear in $eB$

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-}(B) - \Delta E^{\beta,\mu}_{e^+e^-}(0) \simeq\; & \frac{\alpha}{4\pi}\frac{eB}{E_0}\int_{-\infty}^{\infty} dk_0\, \frac{\Theta(k_0^2-m^2)}{\sqrt{k_0^2-m^2}} f_F(k_0)\frac{1}{p_z^2} \\ \times & \left[\frac{\sqrt{k_0^2-m^2}}{p_z}(2E_0k_0 - m^2)\ln\left(\frac{E_0k_0 - m^2 + p_z\sqrt{k_0^2-m^2}}{E_0k_0 - m^2 - p_z\sqrt{k_0^2-m^2}}\right)\right. \\ & \left. -2\,\frac{(k_0^2-m^2)(2k_0^2 - 3E_0k_0 + m^2) + p_z^2(k_0^2 + p_z^2 - E_0k_0)}{(k_0-E_0)^2}\right] \quad . \qquad (4.41)\end{aligned}$$

This expression has an ostensible singularity at $k_0 = E_0$. When expanding around this value we find that the possible divergences cancel, and that the integral is finite as it stands for $p_z > 0$. The limit $p_z \to 0$ is considered in Section 6.

The thermal photon contribution is obtained in a similar manner. We may in this case also perform the $k_0$ integrals. In the field-free case we obtain the well-known result [29–32]

$$\Delta E^{\beta}_{\gamma}(B=0) = \frac{m}{E_0}\,\alpha\frac{\pi}{3}\frac{T^2}{m^2} \quad . \qquad (4.42)$$

Similarly we find the term linear in $eB$

$$\Delta E^{\beta}_{\gamma}(B) - \Delta E^{\beta}_{\gamma}(0) \simeq \frac{m^2}{E_0}\,\frac{\alpha\pi}{6}\frac{T^2}{m^2}\frac{eB}{p_z^2}\left[\frac{E_0}{p_z}\ln\left(\frac{E_0+p_z}{E_0-p_z}\right) - 2\right] \quad . \qquad (4.43)$$

However, these results only apply to an external electron in the lowest Landau level its energy being given according to the tree-level Dirac equation. In the next subsection we shall consider the general case.

## 4.2 Schwinger's method

In the weak-field limit the Landau levels get closer and closer, and in the Furry picture an explicit method of resumming them is always necessary. This resummation was performed in Section 2, and the result expanded to linear order in the external field in the preceding section. However, this problem does not occur when using the Schwinger proper-time method, since no explicit reference to the Landau levels is made. In fact, with Schwinger's method we have an expression in terms of operators which we can apply to any state we wish. We shall now use this method to study the weak-field limit of the self-energy. Starting from Eq. (3.32) we find ($\mathbf{\Pi}^2 = \Pi_\perp^2 + p_z^2$)

$$\begin{aligned}
\hat{\Sigma}_\gamma^\beta &= -\frac{ie^2}{4\pi^{3/2}} \int_{-\infty}^{\infty} [dk_0] \int_{-\infty}^{\infty} dt \int_0^\infty ds f_B(k_0) \frac{1}{(i(s+t))^{3/2}} \\
&\quad \times \exp\left[i\left(tk_0^2 + s(p_0-k_0)^2 - \frac{st}{s+t}\mathbf{\Pi}^2 - m^2 s\right)\right] \\
&\quad \times \left\{e^{-ieBs\sigma_z}\left(\gamma_0(p_0-k_0) - \frac{t}{s+t}\gamma_z p_z\right) - 2m - \frac{t}{s+t} e^{i\frac{st}{s+t}eB\sigma_z} \not{\Pi}_\perp\right\} , \quad (4.44)
\end{aligned}$$

after expanding most terms to linear order in $B$ (the terms with poles as a function of $s$). We keep the full $B$ dependence wherever it is added linearly to $\mathbf{\Pi}^2$ since $\mathbf{\Pi}^2 = p_z^2 + eB(2n+1)$ and $n$ need not be small. In all other places the dependence is $\mathcal{O}(B^2)$. The $t$ and $s$ integrals in Eq. (4.44) can be performed using

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dt}{(it)^{3/2}} \exp\left[i(ta + \frac{b}{t})\right] &= 2\left(\frac{\pi}{b}\right)^{1/2} \sin\left(2\sqrt{ab}\right)\Theta(a) , \\
\int_{-\infty}^{\infty} \frac{dt}{(it)^{3/2}} \frac{1}{t} \exp\left[i(ta + \frac{b}{t})\right] &= 2i\frac{(a\pi)^{1/2}}{b}\left(\frac{\sin\left(2\sqrt{ab}\right)}{2\sqrt{ab}} - \cos\left(2\sqrt{ab}\right)\right)\Theta(a) , \quad (4.45) \\
\int_{-\infty}^{\infty} \frac{dt}{(it)^{3/2}} t \exp\left[i(ta + \frac{b}{t})\right] &= -2i\left(\frac{\pi}{a}\right)^{1/2} \cos\left(2\sqrt{ab}\right)\Theta(a) ,
\end{aligned}$$

for $b > 0$, and

$$\begin{aligned}
i\int_0^\infty dse^{isa} \sin(bs)\frac{1}{s} &= \frac{1}{2}\ln\left|\frac{a-b}{a+b}\right| , \\
i\int_0^\infty dse^{isa} \sin(bs) i &= \frac{b}{a^2-b^2} , \\
i\int_0^\infty dse^{isa}\left(\frac{\sin(bs)}{bs} - \cos(bs)\right)\frac{i}{s} &= -1 - \frac{a}{2b}\ln\left|\frac{a-b}{a+b}\right| , \quad (4.46) \\
i\int_0^\infty dse^{isa}\left(\frac{\sin(bs)}{bs} - \cos(bs)\right)\frac{i}{s}is &= -\frac{1}{2b}\ln\left|\frac{a-b}{a+b}\right| - \frac{a}{a^2-b^2} .
\end{aligned}$$

The final result for the photon contribution is ($k = |k_0|$, $\not{\mathbf{\Pi}} = \gamma\Pi_\perp + \gamma_z p_z$)

$$\hat{\Sigma}^\beta_\gamma = -\frac{e^2}{2\pi}\int_{-\infty}^{\infty}[dk_0]f_B(k_0)\left\{\frac{1}{2\tilde{\mathbf{\Pi}}}\ln\left|\frac{\tilde{a}-\tilde{b}}{\tilde{a}+\tilde{b}}\right|[\gamma_0(p_0-k_0)-2m-\not{\mathbf{\Pi}}]\right.$$
$$\left. -\frac{k}{\tilde{\mathbf{\Pi}}^2}\left(1+\frac{\tilde{a}}{2\tilde{b}}\ln\left|\frac{\tilde{a}-\tilde{b}}{\tilde{a}+\tilde{b}}\right|\right)\not{\mathbf{\Pi}}\right\}\ , \tag{4.47}$$

where ($\tilde{\mathbf{\Pi}} = \sqrt{\tilde{\mathbf{\Pi}}^2}$)

$$\begin{cases} \tilde{a} &= p_0^2 - \tilde{\mathbf{\Pi}}^2 - \tilde{m}^2 - 2p_0k_0\ , \\ \tilde{b} &= 2\tilde{\mathbf{\Pi}}k\ . \end{cases} \tag{4.48}$$

The meaning of $\tilde{m}^2$ and $\tilde{\mathbf{\Pi}}^2$ depends on which $\gamma$-matrix they multiply, and they should be replaced according to

$$\tilde{m}^2 = \begin{cases} m^2\ , & \text{when multiplying } \mathbb{1}, \gamma_x, \gamma_y\ , \\ m^2 + eB\sigma_z\ , & \text{when multiplying } \gamma_0, \gamma_z\ , \end{cases} \tag{4.49}$$

$$\tilde{\mathbf{\Pi}}^2 = \begin{cases} \mathbf{\Pi}^2\ , & \text{when multiplying } \mathbb{1}, \gamma_0, \gamma_z\ , \\ \mathbf{\Pi}^2 - eB\sigma_z\ , & \text{when multiplying } \gamma_x, \gamma_y\ . \end{cases} \tag{4.50}$$

These are just replacement rules that simplify the writing and not any mathematical identities. The Equation 4.47 equals exactly the self-energy in the absence of the magnetic field, with a replacement $p_\mu \to \Pi_\mu$, except that $m^2$ and $\mathbf{\Pi}^2$ should be shifted by $\pm eB\sigma_z$ according to the rules in Eq. (4.49). We write it in this short-hand way in order to more easily see that it reduces to the well-known result in the limit of vanishing $B$. At the same time we notice that it is not enough to simply replace $p_\mu$ by $\Pi_\mu$, but there are some extra $\pm eB\sigma_z$ terms that enter. The electron contribution is calculated in a similar way:

$$\hat{\Sigma}^\beta_{e^+e^-} = \frac{e^2}{2\pi}\int_{-\infty}^{\infty}[dk_0]f_F(k_0)\Theta(k_0^2-\tilde{m}^2)$$
$$\times\left\{\frac{1}{2\tilde{\mathbf{\Pi}}}\ln\left|\frac{\tilde{a}-\tilde{b}}{\tilde{a}+\tilde{b}}\right|[\gamma_0k_0-2m]+\frac{\tilde{k}}{\tilde{\mathbf{\Pi}}^2}\left(1+\frac{\tilde{a}}{2\tilde{b}}\ln\left|\frac{\tilde{a}-\tilde{b}}{\tilde{a}+\tilde{b}}\right|\right)\not{\mathbf{\Pi}}\right\}\ , \tag{4.51}$$

where now $\tilde{k} = \sqrt{k_0^2 - \tilde{m}^2}$ and

$$\begin{cases} \tilde{a} &= p_0^2 - \tilde{\mathbf{\Pi}}^2 + \tilde{m}^2 - 2p_0k_0\ , \\ \tilde{b} &= 2\tilde{\mathbf{\Pi}}\tilde{k}\ . \end{cases} \tag{4.52}$$

Again, it is easy to check that the zero-field limit from Eq. (4.51) agrees with standard results.

One advantage with Eqs. (4.47, 4.51) is that they are expressed directly in terms of gauge-invariant operators and they can be used to calculate the expectation value between

any Landau levels. It is particularly useful in Section 5 where we solve a self-consistent dispersion relation without specifying the exact form of the spinors. On the other hand, the results agree with the ones from the Furry picture where we have checked them. An example is the anomalous magnetic moment that we discuss in Section 6.

# 5 High-temperature limit

Since a few years back there has been a great interest in the high-temperature limit of gauge theories, mainly stimulated by the successful resummation of a consistent and infinite set of diagrams, which is encoded in the so-called Hard Thermal Loop (HTL) [19, 18] effective action. In the high-temperature limit it is meaningful to not only compute a perturbative correction to the energy spectrum, but also to solve the dispersion relation self-consistently. The reason is that in QED the dominating $\mathcal{O}(eT)$ correction comes only from the one-loop diagram that we have calculated, all higher-order diagrams being suppressed by extra factors of $\mathcal{O}(e^2T)$. The leading terms are also gauge-fixing independent and gauge-invariant, which makes the whole procedure consistent. Therefore, it is particularly interesting to take the high-temperature limit ($T \gg m,\ p_z,\ \sqrt{eB},\ \mu$) and study the effects of a weak magnetic field. The effective Dirac equation is

$$\begin{aligned} &[\not{\Pi} - m - \hat{\Sigma}(p_0, p_z, \mathbf{\Pi}_\perp)]\Psi = \\ &\Big[s(p_0, \mathbf{\Pi}^2)\gamma_0 p_0 - r(p_0, \mathbf{\Pi}^2)\gamma_z p_z - r(p_0, \mathbf{\Pi}^2 - eB\sigma_z)\not{\Pi}_\perp - m\Big]\Psi = 0 \ , \end{aligned} \tag{5.53}$$

where as before $\mathbf{\Pi}^2 = \Pi_\perp^2 + p_z^2$ and

$$\begin{aligned} s(p_0, \mathbf{\Pi}^2) &= \left(1 - \frac{\mathcal{M}^2}{2p_0\,|\mathbf{\Pi}|}\ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right|\right) \ , \\ r(p_0, \mathbf{\Pi}^2) &= \left(1 + \frac{\mathcal{M}^2}{\mathbf{\Pi}^2}\left(1 - \frac{p_0}{2\,|\mathbf{\Pi}|}\ln\left|\frac{p_0 + |\mathbf{\Pi}|}{p_0 - |\mathbf{\Pi}|}\right|\right)\right) \ . \end{aligned} \tag{5.54}$$

The temperature dependence enters only through the thermal mass $\mathcal{M}^2 = e^2T^2/8$. It is almost possible to guess the expression in Eq. (5.53) from the standard expression for the HTL Dirac equation. The usual momentum $p_\mu$ should be replaced with the gauge-invariant momentum $\Pi_\mu$, but there is an ambiguity in replacing $p^2$ by $\Pi^2$ or by $\not{\Pi}\not{\Pi}$. The correct way follows from the lengthy calculations in Section 4. It would also be interesting to compare Eq. (5.53) with the equation of motion obtained directly from the HTL effective

action. This is in general difficult due to the non-local and non-linear character of the HTL effective action, but we have checked that it agrees up to linear order in $B$.

From now on we shall take the high-temperature limit and neglect the vacuum mass, which simplifies the Dirac equation considerably. The matrix structure of the function $r(p_0, \mathbf{\Pi}^2 - eB\sigma_z)$ can be made explicit by rewriting it as

$$r(p_0, \mathbf{\Pi}^2 - eB\sigma_z) = \frac{1}{2}(\mathbb{1} + \sigma_z) r(p_0, \mathbf{\Pi}^2 - eB) + \frac{1}{2}(\mathbb{1} - \sigma_z) r(p_0, \mathbf{\Pi}^2 + eB) \ . \tag{5.55}$$

In the chiral representation Eq. (A.1) we obtain the following Dirac equation

$$\begin{pmatrix} 0 & 0 & -p_0 s - p_z r & -r_- \xi_+ \\ 0 & 0 & -r_+ \xi_- & -p_0 s + p_z r \\ -p_0 s + p_z r & r_- \xi_+ & 0 & 0 \\ r_+ \xi_- & -p_0 s - p_z r & 0 & 0 \end{pmatrix} \Psi = 0 \ , \tag{5.56}$$

where $\xi_\pm = \Pi_x \mp i \Pi_y$ and $r_\pm = r(p_0, \mathbf{\Pi}^2 \pm eB)$. Using the ansatz in Eq. (3.35), the relations in Eq. (A.8), and dividing $\Psi$ into left and right 2-component spinors ($u = (R,\, L)^T$ in the conventions of [22]), we find the decomposed Dirac equations

$$\begin{aligned} \begin{pmatrix} -E s_n - p_z r_n & -i\sqrt{2eBn}\, r_{n-\frac{1}{2}} \\ i\sqrt{2eBn}\, r_{n-\frac{1}{2}} & -E s_{n-1} + p_z r_{n-1} \end{pmatrix} L &= 0 \ , \\ \begin{pmatrix} -E s_n + p_z r_n & i\sqrt{2eBn}\, r_{n-\frac{1}{2}} \\ -i\sqrt{2eBn}\, r_{n-\frac{1}{2}} & -E s_{n-1} - p_z r_{n-1} \end{pmatrix} R &= 0 \ , \end{aligned} \tag{5.57}$$

where $s_n = s(E, p_z^2 + eB(2n+1))$, and similarly for $r_n$. The dispersion relations follow immediately by taking the determinants of Eq. (5.57)

$$\begin{aligned} L: &\quad (E s_n + p_z r_n)(E s_{n-1} - p_z r_{n-1}) - 2eBn r_{n-\frac{1}{2}}^2 = 0 \ , \\ R: &\quad (E s_n - p_z r_n)(E s_{n-1} + p_z r_{n-1}) - 2eBn r_{n-\frac{1}{2}}^2 = 0 \ . \end{aligned} \tag{5.58}$$

These relations are valid for all $n \geq 1$, but in the lowest Landau level there is only one non-zero component for each of $L$ and $R$, so the dispersion relations reduce to

$$\begin{aligned} L: &\quad E s(E, p_z^2 + eB) + p_z r(E, p_z^2 + eB) = 0 \ , \\ R: &\quad E s(E, p_z^2 + eB) - p_z r(E, p_z^2 + eB) = 0 \ . \end{aligned} \tag{5.59}$$

Since the high-temperature correction is a consistently resummed large correction to the vacuum dispersion relations, we want to solve them self-consistently. In the absence of a

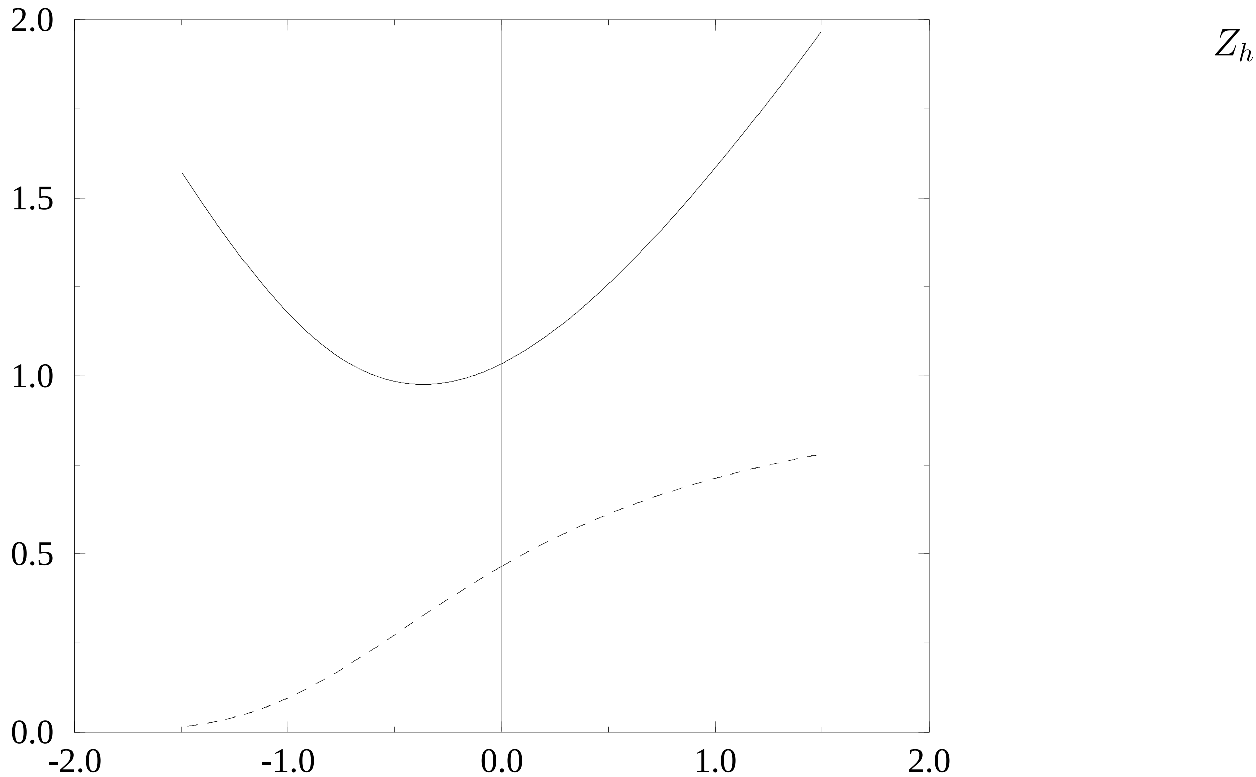


**Figure 1**: *Dispersion relation and spectral weight for the right-handed branch in the lowest Landau level ($n = 0$). All dimensionful parameters are given in units of the thermal mass $\mathcal{M}$.*

magnetic field it is well known that the high-temperature dispersion relation for fermions has two branches [15, 16, 17] and that spin up/down (or left/right handedness) is degenerate. The hole branch corresponds to a positive energy solution to the factor in the dispersion relation that usually only gives a negative energy solution. We have a similar phenomenon here, and the hole solution is associated with the factor $Es_n + p_z r_n$ for positive $p_z$, even though the dispersion relation does not factorize completely. There are almost the separate symmetries ($p_z \leftrightarrow -p_z$), ($E \leftrightarrow -E$) and ($L \leftrightarrow R$), but they fail because of the difference in the index $n$ for the two factors in Eq. (5.58), which also breaks the spin degeneracy. We notice on the other hand that there *are* symmetries under the combinations ($L \leftrightarrow R$, $p_z \leftrightarrow -p_z$), ($L \leftrightarrow R, E \leftrightarrow -E$) and ($E \leftrightarrow -E, p_z \leftrightarrow -p_z$). The number of states for given values of $p_z$ and $n$ is eight, corresponding to $(L/R)\times$ (particle/hole) $\times$ (positive/negative energy), apart from the usual degeneracy in $p_y$. In the lowest Landau level, only right-handed particles and left-handed holes can propagate for

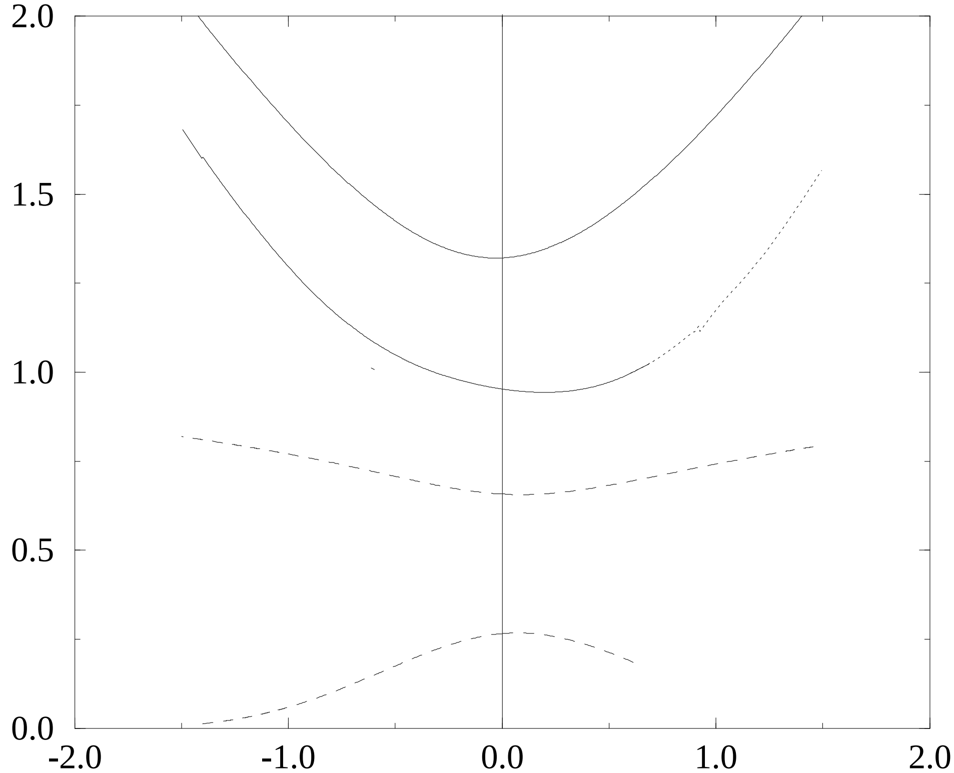


**Figure 2**: *Dispersion relations and spectral weights for the next-to-lowest Landau level ($n = 1$). The dotted part of $\omega_h$ shows a continuation of the dispersion relation beyond the point where $\omega_h$ picks up an imaginary part. All dimensionful parameters are given in units of the thermal mass $\mathcal{M}$.*

positive $p_z$, and vice versa for negative $p_z$ (see Eq. (5.59)), as can be understood from the following argument. The energy shift for a magnetic moment $\boldsymbol{\mu}$ is given by $-\boldsymbol{\mu} \cdot \mathbf{B}$. The electron-spin contribution to $\boldsymbol{\mu}$ points opposite to the spin, since the charge is negative. We have chosen $\mathbf{B}$ to point in the negative $z$-direction so that the energy is lowest when the spin points in the positive $z$-direction. For $p_z > 0$ the helicity should thus be positive in the lowest Landau level and we expect the positive chirality state $R$ to propagate (see Fig. 1). Left-handed particles have the wrong helicity to be in the lowest Landau level for $p_z > 0$. However, holes have opposite chirality–helicity relation, so a left-handed hole can propagate for $p_z > 0$. Similarly, right-handed holes propagate for $p_z < 0$ as shown in Fig. 1. A similar asymmetry exists in higher Landau levels in the sense that, for a right-handed particle with momentum $p_z$ and $\mathbf{\Pi}^2 = p_z^2 + eB(2n+1)$, there exists another right-handed particle state with momentum $-p_z$ but with $\mathbf{\Pi}^2 = p_z^2 + eB(2n-1)$ and a different energy.

It is clear that this asymmetry can be important for decay processes where only one chirality is produced, for instance in $\beta$-decay. To get an asymmetry of this form, we need a chiral charged particle with spin (or magnetic moment). It clearly needs to have a splitting between spin up and down, but it is also important that it is chiral in order to separate right- and left-handed particles at the same time. At zero temperature there are no chiral charged spin 1/2 particles in the Standard Model, but at high temperature the dominant mass effect is the thermal one, which is chirally invariant. In this way the temperature effect makes the electron essentially chiral without reducing its magnetic moment.

It has been observed [33, 34] that the polarization of electron and positron spins in a strong magnetic field generates an anisotropy of the neutrino emission in weak processes, and that this anisotropy could be the explanation of the high space velocities of pulsars. In the presence of a thermal heat bath, we have seen above that a further left–right asymmetry is generated for the electrons and positrons. It remains to be seen to what extent this asymmetry increases the asymmetry in the neutrino emission during the hot phase of the supernova explosion.

In order to find out the correct final-state factors for a decay into the different branches, not only the spectrum is needed but also the spectral weight for each branch. That is, we need the wave function renormalization factor $Z(k)$. It can be computed from [17]

$$Z_i(p_z, n)^{-1} = \frac{d}{d\omega}\bigg|_{\omega=E_i(p_z,n)} \left(\mathrm{Tr}\left[(\mathcal{D}(\omega, p_z, n)\gamma_0)^{-1}\right]\right)^{-1} , \tag{5.60}$$

where $\mathcal{D}(\omega, p_z, n)$ is the matrix in Eq. (5.56) and $E_i$ corresponds to the solutions for $L$, $R$, particle and hole. The spectral functions should satisfy the sum rule for $n \geq 1$

$$Z_{Lp} + Z_{Lh} + Z_{Rp} + Z_{Rh} + \text{multiparticle states} = 2 . \tag{5.61}$$

We use a definition of $Z$ that differs from [17] by a factor of 2. In the lowest Landau level there are only half as many states, so the spectral weights add up to 1. The result of a numerical calculation of the spectral weight as well as of the spectrum and for the lowest Landau level is presented in Fig. 1. It shows the right-handed branch, but the picture is the same for the left-handed branch with $p_z \leftrightarrow -p_z$. A similar plot for $n = 1$ is given in Fig. 2. When solving Eq. (5.58) it turns out that there can be an imaginary part of $E_h$ on-shell for certain values of $p_z$. Let us take the $R$ branch as an example. For $p_z \gg \mathcal{M}$ the hole branch can exist if $Es_{n-1} + p_z r_{n-1}$ does not grow with $p_z$, and that is possible since

$s(E, p_z^2 + eB(2n-1))$ can be negative when the logarithm in Eq. (5.54) dominates, i.e. $E \gtrsim \sqrt{p_z^2 + eB(2n-1)}$. The larger $p_z$ is, the closer $E$ is to the above square root. This is why the hole branch goes exponentially close to the light cone for large $p_z$ in the standard analysis [17]. Here we also have a factor $(Es_n - p_z r_n)$ in the right dispersion relation in Eq. (5.58). In $s_n$ and $r_n$ the argument in the logarithms contains $E - \sqrt{p^2 + eB(2n+1)}$. It is then clear that the energy can go below the $n$-th light cone when $p_z$ is large enough and the logarithm picks up an imaginary part. The whole approximation breaks down at these points since the excitations are no longer stable. The value of the spectral function does not satisfy Eq. (5.61) when this happens and it is not clear to what extent it is meaningful to continue beyond these points, or even to go close to them.

# 6 Energy in the lowest Landau level – Anomalous magnetic moment

The anomalous magnetic moment of electrons is defined from the energy spectrum of a particle nearly at rest in a weak field. Usually, it is calculated from the triangle diagram (see Fig. 3), using plane waves as external states, while it really should be done in the proper Landau levels as explained in Section 4. Furthermore, not only is the shift of one level required but also the energy gap between different levels generated by the field. In vacuum the anomalous energy shift is independent of the Landau level for low enough fields (see e.g. [14]). We have not yet calculated the thermal shift for an arbitrary Landau level, but there are some interesting features already in the lowest Landau level. In vacuum one can calculate the anomalous magnetic moment either from a triangle diagram, using plane waves as external states, or extract it from the linear term in the exact expression with the electrons in the general Landau levels [21]. The result is the same. At finite temperature the issue is more complicated because of the IR sensitivity and it really makes a difference if the external states are in a proper Landau level, or plane-wave states. A related problem was discovered in [35, 36], where a real-time method was used for the triangle diagram, but we later found that the limit procedure of taking the external photon momentum to zero is not unique and thus the result is not well defined. We shall exemplify this phenomenon here by calculating the anomalous magnetic moment both from the expressions in Section 4 and from the triangle diagram (Section 7.1) in the static limit.

# 7 The Self-energy method

Comparing with the energy shift for a charge $-e$, particle with spin ($\mathbf{s}$) magnetic moment $\boldsymbol{\mu} = -e/2m\, g\mathbf{s}$ in a magnetic field $\mathbf{B} = -B\mathbf{e}_z$, $\Delta E = -\boldsymbol{\mu}\cdot\mathbf{B}$, we can write for an electron in the lowest Landau level ( here $\langle s_z\rangle = 1/2$):

$$\Delta E \equiv \langle \Psi_{1;0,0,0}|\hat{\Sigma}|\Psi_{1;0,0,0}\rangle \equiv \Delta m^{\beta,\mu} - \frac{eB}{2m}\frac{\delta g}{2} + \mathcal{O}(eB)^2 \quad , \tag{7.62}$$

where $\Delta m^{\beta,\mu}$ is the thermal mass, and $\delta g/2$ is the anomaly of the magnetic moment. To linear order in the magnetic field, Eq. (2.13) gives the very well-known result [37, 21, 12]

$$\Delta E_{\text{vac}} = -\frac{eB}{2m}\frac{\alpha}{2\pi} \quad . \tag{7.63}$$

The total energy remains positive even for very large $B$ when the non-linear terms dominate [38]. Let us now consider the thermal photon contribution using Schwinger's proper-time method. When the operators in Eq. (4.47) act on a Landau level, we can use the properties (see Appendix A)

$$\begin{aligned}
\Pi_\perp^2 \Psi^{(\pm)}_{\zeta;n,p_y,p_z} &= eB[2n+\sigma_z]\Psi^{(\pm)}_{\zeta;n,p_y,p_z} \ , \\
\sigma_3 \Psi^{(\pm)}_{1;0,0,0} &= \Psi^{(\pm)}_{1;0,0,0} \ , \\
\not{\Pi}_\perp \Psi^{(\pm)}_{1;0,0,0} &= 0 \ ,
\end{aligned} \tag{7.64}$$

to compute the expectation value of $\hat{\Sigma}^\beta_\gamma$ in the lowest Landau level. In the limit of vanishing $p_z$ we obtain to linear order in $eB$

$$\Delta E^\beta_\gamma(p_z=0) = m\left(\frac{\alpha\pi}{3}\frac{T^2}{m^2} + \frac{eB}{2m^2}\frac{2\alpha\pi}{9}\frac{T^2}{m^2}\right) \quad , \tag{7.65}$$

which agrees with standard results [29–32], and also with the Furry picture in Eq. (4.42) and the limit $p_z \to 0$ in Eq. (4.43). We also notice that the main contribution comes from the hard part of the loop integral ($k \simeq T$) and there is no particular IR sensitivity.

For thermal electrons we similarly obtain in the limit of vanishing $B$ and $p_z$:

$$\begin{aligned}
&\Delta E^{\beta,\mu}_{e^+e^-}(B=0,p_z=0) \equiv \Delta m^{\beta,\mu}_{e^+e^-} \\
&= \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} dk_0 \Theta(k_0^2-m^2) f_F(k_0)\sqrt{k_0^2-m^2}\,\frac{k_0-2m}{m(k_0-m)} \quad ,
\end{aligned} \tag{7.66}$$

which agrees with the corresponding limit of Eq. (4.39). The thermal-electron contribution to linear order in the magnetic field is more involved. First, there is a $B$ dependence in the

$\Theta$-function in Eq. (4.51), which after expansion generates a derivative of the distribution function. Furthermore, one cannot expand the integrand in $B$ naively, because that would generate IR-divergent integrals. Another delicate integration by parts of the logarithm in Eq. (4.51) is needed to give a finite integral after the expansion. There is thus an ambiguity in the way of writing the self-energy, depending on which terms are integrated by parts to produce derivatives of the distribution function. We have chosen to perform further integrations by parts to remove the derivative on the distribution function whenever this does not produce any IR divergences. After these manipulations we find, up to linear order in $B$:

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-} &= \Delta m^{\beta,\mu}_{e^+e^-} + \frac{eB}{2m}\frac{\alpha}{3\pi}\int_m^\infty \frac{d\omega}{\sqrt{\omega^2-m^2}} \\ &\quad \times \left[\left(\frac{2\omega^2+2m\omega-m^2}{m^2} - \frac{m}{(\omega+m)}\right) f_F^+(\omega) + 2m\frac{df_F^+(\omega)}{d\omega}\right] + \\ &\quad + \frac{eB}{2m}\frac{\alpha}{3\pi}\int_m^\infty \frac{d\omega}{\sqrt{\omega^2-m^2}} \frac{2\omega^3-3m^2\omega-2m^3}{m^2(\omega+m)} f_F^-(\omega) \quad . \end{aligned} \tag{7.67}$$

In order to obtain this result using the Furry picture propagator, we must take the limit $p_z \to 0$ in Eq. (4.41). Obviously the expansion will be in powers of $p_z/(k_0 - m)$. This produces IR singularities, if we are not careful. For the antiparticle part $-k_0 = \omega > 0$, so here it is straightforward to perform the expansion. For the particle part we know, from above and by comparison with results using the imaginary-time formalism, that it is likely that the finite result should contain derivatives of the distribution function. Let us therefore introduce the factor of convergence $(\omega - m)^\nu$, where $\nu$ is assumed to be so large that we may perform the expansion to get the contribution for vanishing $p_z$. We may then isolate the term that will become divergent as $\nu \to 0$. Integrating this term by parts, the out-integrated term vanishes for $\nu > 1$. In the end we may consider the analytical continuation $\nu \to 0$, and arrive at Eq. (7.67).

In the high-temperature limit we obtain

$$\Delta E^{\beta,\mu}_{e^+e^-} \simeq m\left(\frac{\alpha\pi}{6}\frac{T^2}{m^2} + \frac{eB}{2m^2}\frac{\alpha\pi}{9}\frac{T^2}{m^2}\right). \tag{7.68}$$

Again, it agrees with standard results for the high-temperature limit and it is dominated by a hard thermal loop. The difference with Eq. (7.65) is that Eq. (7.68) is only approximate in the high-temperature limit and that sub-leading terms *are* IR-sensitive. This is why we had to be so careful with the expansion in $B$.

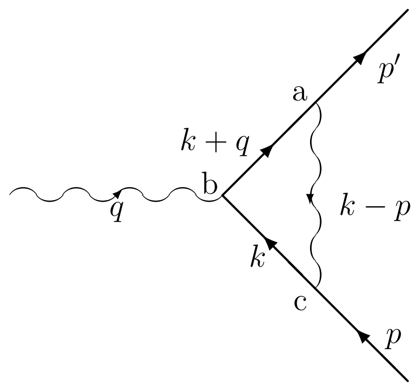


**Figure 3**: *The triangle diagram.*

## 7.1 The triangle diagram

The traditional way of computing the anomalous magnetic moment is from the triangle diagram in Fig. 3. In vacuum, the full vertex, sandwiched between plane-wave states, can be written as

$$\begin{aligned}\bar{u}_{p'}\Gamma_\mu(p',p)u_p &= \bar{u}_{p'}\left[F_1(q^2)\gamma_\mu + F_2(q^2)\frac{i\sigma_{\mu\nu}q^\nu}{2m}\right]u_p \\ &= \bar{u}_{p'}\left[\Big(F_1(q^2)+F_2(q^2)\Big)\gamma_\mu - F_2(q^2)\frac{p'_\mu + p_\mu}{2m}\right]u_p\ , \qquad (7.69)\end{aligned}$$

where the Gordon decomposition was used in the last equality. The anomalous magnetic moment is defined by $\delta g/2 = F_2(0)$ and it can be extracted from the term linear in $p_\mu$ in the limit $q \to 0$. When using the Gordon decomposition, it should be kept in mind that the external states are on the $B = 0$ mass shell. The use of such states is questionable from the point of view of perturbation theory, as discussed in the beginning of Section 4.

At finite temperature there are two formalisms for calculating the triangle diagram, the imaginary- and real-time formalisms (ITF and RTF). In the ITF we can put $q_\mu = 0$ from the outset but we are left with an analytic continuation in the $p_0$ variable. On the other hand, in the RTF (we shall use Thermo Field Dynamics (TFD) as a RTF) we get the result for real $p_0$, but it is more tricky to put $q_\mu = 0$ before doing the loop integration, because there are potentially ill-defined products of distributions. There has recently been much interest in two issues that are of importance here. First, the relation between ITF and TFD has been clarified. It was first found that the naive diagrams in TFD (the ones with only physical fields on the external legs) did not give the same result as the ITF in general [39, 40, 41]. The difference was shown to be related to the different kinds of analytic continuations that are possible in the ITF, and different time-ordered, retarded

and advanced Green functions [42, 43, 44]. It is now clear that a certain combination of TFD diagrams gives the same result as the usual analytic continuation ($\omega \to p_0 + i\epsilon$) in ITF, which is the retarded Green function. Secondly, the Lorentz invariance is broken at finite temperature so the vertex function depends on $q_0$ and $\mathbf{q}$ independently. We can thus take the $q_\mu \to 0$ limit with either $q^2 > 0$ or $q^2 < 0$ (time-like and space-like limits) or generally with any fixed ratio $|\mathbf{q}|/q_0$. It has been known for some time that e.g. the one-loop self-energy in the $\phi^3$-theory does not have a unique limit when $q_\mu \to 0$ [45, 46]. In principle it would be interesting to discuss both limits, but the external mass shell conditions on $p$ and $p'$ put the constraint on $q$ that

$$p_0 q_0 - \mathbf{p}\mathbf{q} + \frac{1}{2}(q_0^2 - \mathbf{q}^2) = 0 \ , \tag{7.70}$$

which prevents the use of $q_0$ and $\mathbf{q}$ as independent variables. We shall therefore concentrate on the case with $q \equiv 0$ for both the ITF and RTF.

In the ITF we can simply put $q$ to zero from the outset. We use standard ITF rules in Euclidean space and write the expression for the vertex correction as

$$-ie\Lambda_\mu = (-ie)^3 \int [d^4k] \frac{-i}{(k-p)^2 + i\epsilon} \left( \frac{i}{k^2 - m^2 + i\epsilon} \right)^2 T_\mu(k) \ . \tag{7.71}$$

The simplified tensor structure in the numerator is

$$T_\mu(k) \equiv \gamma^\nu (\not{k} + m) \gamma_\mu (\not{k} + m) \gamma_\nu = 2(k^2 - m^2)\gamma_\mu + 4(2m - \not{k}) k_\mu \ . \tag{7.72}$$

The last term can be replaced by $4(2m - k_0)k_i$, since we are only interested in terms proportional to $p_i$ (and not to $\gamma_i$) and we neglect higher-order terms in $p_i$ in the non-relativistic limit. Then also $\gamma_0$ can be replaced by $\mathbb{1}$. The double pole in $k^2 - m^2$ can conveniently be represented by a derivative,

$$\frac{\partial}{\partial M^2} \frac{1}{k^2 - M^2 + i\epsilon} \bigg|_{M^2 = m^2} = \frac{1}{(k^2 - m^2 + i\epsilon)^2} \ . \tag{7.73}$$

After an analytic continuation we find as usual two contributions, one from thermal photons and one from thermal electrons. The photon contribution agrees with Eq. (7.65) so we leave that aside. From the thermal electrons we obtain

$$\begin{aligned} \frac{\delta g^{\beta,\mu}_{e^+e^-}}{2} &= -\frac{\alpha}{3\pi} \int_m^\infty d\omega (\omega^2 - m^2)^{3/2} \left\{ \frac{2}{m^2\omega^2} (f_F^+(\omega) + f_F^-(\omega)) \right. \\ &\quad \left. - \left[ \frac{df_F^+(\omega)}{d\omega} \frac{2m - \omega}{m\omega(m-\omega)^2} + \frac{df_F^-(\omega)}{d\omega} \frac{2m + \omega}{m\omega(m+\omega)^2} \right] \right\} \ . \end{aligned} \tag{7.74}$$

We would like to remark here that special care has to be taken when splitting into matter and vacuum contributions, when the Feynman diagram considered contains fermion as well as boson propagators. In the so-called finite-density contribution (see e.g. Eq. (3.71) in [47]), there is a term $\Theta(p_0 - \omega)$ that could have been mistaken for $\Theta(\mu - \omega)$. This term has to be added to the vacuum contribution in order to give the correct Feynman pole prescription for the vacuum part of the fermion propagator. The leading high-temperature contribution comes from the highest power of $\omega$ in the integral and it is given by

$$\frac{\delta g^{\beta,\mu}_{e^+e^-}}{2} \simeq -\frac{\alpha\pi}{9}\left(\frac{T}{m}\right)^2 \ , \tag{7.75}$$

which agrees with the result in Eq. (7.68). The sub-leading terms, which are sensitive to the IR part of the loop integral, are on the other hand completely different.

In TFD there are several different vertex functions depending on how the external legs are chosen to be particle or thermal ghost lines and we call them $\Lambda_{abc}$ $(a, b, c = 1, 2)$. According to [43, 48, 44] it is necessary to sum over 1 and 2 with certain weights for all external points except one. The external point with only a 1-field is the one with the latest time argument. In our case we consider an incoming electron going through the heat-bath and interacting with an external magnetic field. The incoming point $c$ in Fig. 3 must then have earlier time than point $a$. Also, the external field at point $b$ does not influence the electron at later times than the time at point $a$. We are thus naturally led to consider the combination with $a$ as the latest time. According to [43] the retarded Green function with the outgoing electron at the latest time and $q = 0$ is given by

$$\Lambda = (\Lambda_{111} + \Lambda_{121}) + e^{-\beta p_0/2}(\Lambda_{112} + \Lambda_{122}) \ . \tag{7.76}$$

In Eq. (7.76) there are several terms with overlapping $\delta$-functions for $q_\mu = 0$ but they actually add up to give something finite. In particular we see that the incoming photon line at point $b$ should be summed over 1 and 2 with the consequence that there is no product of $\delta$-functions from the two fermion propagators. The $2 \times 2$ structure of the matrix propagator from $c$ to $a$ becomes

$$- T_\mu(k) \begin{pmatrix} G^2(k) - \sin^2\vartheta[G^2(k) - G^{*2}(k)] & \sin\vartheta\cos\vartheta[(G^2(k) - G^{*2}(k)] \\ -\sin\vartheta\cos\vartheta[(G^2(k) - G^{*2}(k)] & G^{*2}(k) - \sin^2\vartheta[G^2(k) - G^{*2}(k)] \end{pmatrix} \ , \tag{7.77}$$

where $G(k) = (k^2 - m^2 + i\epsilon)^{-1}$ and $\sin^2\vartheta = f_F(k_0)$. The combination $G^2(k) - G^{*2}(k)$ turns

out to be related to a derivative of a $\delta$-function,

$$G^2(k) - G^{*2}(k) = 2\pi i \left.\frac{\partial}{\partial M^2}\right|_{M^2=m^2} \delta(k^2 - M^2) , \tag{7.78}$$

which eventually leads to exactly the same expression as in the ITF. The remaining multiple $\delta$-functions do not contribute for on-shell external particles when $q_\mu = 0$. The only thermal contributions are linear in the electron and photon distribution functions. The same form of regularization of the product between a propagator and a $\delta$-function with the same argument was advocated in [49], but here it followed simply from using the rules of [43]. It has been shown for the $n$-point function in a scalar $\phi^3$-theory that the ITF (and the mass-derivative formula) is recovered in the space-like limit but *not* in the time-like [45]. This suggests that putting $q_\mu = 0$ from start corresponds to the space-like limit and the time-like limit could very well give a different result. For a static external field, with space-dependent gauge fields, it seems most appropriate to take the space-like limit. Nevertheless, we do not find agreement between the triangle diagram Eq. (7.74) and $\delta g^{\beta;\mu}_{e^+e^-}$ inferred from Eq. (7.67), apart from the leading high-temperature terms. We surmise that the reason for this is that perturbation theory in $B$ is not adequate, as discussed at the beginning of this section, and that the full Landau levels should be used as external states.

# 8 The strong-field limit

In the limit of extremely strong magnetic fields, the energies in all but the lowest Landau level are approximately proportional to $\sqrt{eB}$. This implies that intermediate states of higher Landau levels are suppressed. It is thus very convenient to use the Furry picture propagator with its explicit spectral decomposition in this case. One may then neglect the contributions from all but the lowest Landau level, i.e. the sum over $n$ is reduced to $n = 0$. In Section 8.1 we calculate the self-energy in this approximation. In Section 8.2 we explicitly verify that in this limit of strong magnetic fields the self-energy is independent of the gauge-fixing parameter in the photon propagator.

## 8.1 The self-energy in strong fields

In [12] it was shown that the approximation of only considering intermediate electrons in the lowest Landau level gives the same result as the high-field limit of Eq. (2.13), i.e.

$$\Delta E_{\rm vac} \simeq \frac{m^2}{E_0}\frac{\alpha}{4\pi}\left(\ln\frac{2eB}{m^2}\right)^2 + \mathcal{O}\left(\ln\frac{2eB}{m^2}\right) \quad , \quad eB \gg m^2, p_z^2 \quad . \tag{8.79}$$

When the contribution from thermal electrons is considered, it is perfectly clear from the suppression by the distribution functions that the lowest Landau level is dominating for $eB \gg m^2, p_z^2, \mu^2, T^2$. After reducing the sum to $n = 0$ in Eq. (2.14) and performing the integrals as before, we may now also perform the remaining Gaussian integrals over $x, x'$. The result reads

$$\Delta E^{\beta,\mu}_{e^+e^-} \simeq \frac{m^2}{E_0}\frac{2\alpha}{\pi^2} eB \int d^4k \frac{f_F(k_0)\delta(k_0^2 - m^2 - k_z^2)}{(k_0 - E_0)^2 - (k_z - p_z)^2 - 2eBk_\perp^2 + i\varepsilon} \exp(-k_\perp^2) \quad . \tag{8.80}$$

Let us now use the $\delta$-function to integrate over $k_z$. Then we perform the angular integral in $\mathbf{k}_\perp$ and substitute $u = k_\perp^2$. We find

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-} \simeq & -\frac{m^2}{E_0}\frac{\alpha}{2\pi}\int_{-\infty}^{\infty} dk_0 \frac{\Theta(k_0^2 - m^2)}{\sqrt{k_0^2 - m^2}} f_F(k_0) \\ & \times \sum_{\pm}\int_0^{\infty} du \frac{\exp(-u)}{u - [(k_0 - E_0)^2 - (p_z \pm \sqrt{k_0^2 - m^2})^2 + i\varepsilon]/2eB} \quad .\end{aligned} \tag{8.81}$$

We may here identify exponential integrals, and use their asymptotic expansions as given in Appendix E. The leading strong-field behaviour is thus

$$\begin{aligned}\Delta E^{\beta,\mu}_{e^+e^-} \quad \simeq \quad & -\frac{m^2}{E_0}\frac{\alpha}{\pi}\int_{-\infty}^{\infty} dk_0 \frac{\Theta(k_0^2 - m^2)}{\sqrt{k_0^2 - m^2}} f_F(k_0) \ln\left(\frac{eB}{m|k_0 - E_0|}\right) \\ & - i\frac{m}{E_0}\alpha \int_m^{\infty} \frac{d\omega}{\sqrt{\omega^2 - m^2}} f_F^-(\omega) \quad .\end{aligned} \tag{8.82}$$

Notice here the field-independent (in this limit) imaginary part, which is exponentially suppressed for weaker fields. We have also investigated the self-energy for a positron in the lowest Landau level. The result is the same as in the electron case above, but $f_F^-(\omega)$ is replaced by $f_F^+(\omega)$ in the imaginary part on the right-hand side of Eq. (8.82). The physical process which is responsible for the occurrence of the imaginary part of $\Delta E^{\beta,\mu}_{e^+e^-}$ is the reaction $e^+ e^- \mapsto \gamma$, which is possible in the presence of an external field that may absorb momentum to make the process energetically possible [50]. We have calculated in Appendix F the tree-level decay rate $\Gamma$, for an electron in a background of positrons. The

result is just as expected, $\Gamma \equiv 2\,\mathrm{Im}\,\Delta E^{\beta,\mu}_{e^+e^-}$. The way we obtain the imaginary part in $\Delta E^{\beta,\mu}_{e^+e^-}$ in the strong-field limit with the $+\varepsilon$ prescription in Eq. (8.81) is similar to the pole prescription of Landau in the theory of longitudinal plasma oscillations (see e.g. [51]).

For the thermal-photon contribution we use instead the $\delta$-function to integrate over $k_\perp$, and find the result corresponding to Eq. (8.81),

$$\Delta E^\beta_\gamma \simeq \frac{m^2}{E_0}\frac{\alpha}{\pi}\int_{-\infty}^{\infty} dk_0 f_B(k_0)\int_{-k+p_z}^{k+p_z} dk_z\,\frac{\exp\{-[k^2-(k_z-p_z)^2]/2eB\}}{k_z^2-(k^2+2E_0k_0+p_z^2)-i\varepsilon}\quad , \tag{8.83}$$

where $k \equiv |k_0|$. We may, to leading order, approximate the exponential function with unity. The poles of the integrand are outside the interval of integration in $k_z$, so the thermal photon contribution is real, and we may let $\varepsilon$ vanish. However, $k_0^2 + 2E_0k_0 + p_z^2$ changes sign at $k_0 = -E_0 \pm m$, so we must split the $k_0$ integral in two parts. After performing the $k_z$-integral we thus obtain

$$\begin{aligned}\Delta E^\beta_\gamma \simeq\ & \frac{m^2}{E_0}\frac{\alpha}{\pi}\int_{E_0-m}^{E_0+m}\frac{dk f_B(k)}{\sqrt{2E_0k-k^2-p_z^2}}\\ &\times\left[\arctan\left(\frac{k+p_z}{\sqrt{2E_0k-k^2-p_z^2}}\right)+\arctan\left(\frac{k-p_z}{\sqrt{2E_0k-k^2-p_z^2}}\right)\right]\\ &-\frac{m^2}{E_0}\frac{\alpha}{2\pi}\left\{\int_{-\infty}^{-E_0-m}+\int_{-E_0+m}^{\infty}\right\}\frac{dk_0 f_B(k_0)}{\sqrt{k^2+2E_0k_0+p_z^2}}\\ &\times\ln\left(\frac{k^2+E_0k_0+k\sqrt{k^2+2E_0k_0+p_z^2}}{k^2+E_0k_0-k\sqrt{k^2+2E_0k_0+p_z^2}}\right).\end{aligned} \tag{8.84}$$

Notice that this is finite since there are cancellations between all of the ostensible IR divergences. For $p_z^2 > 0$, $k_0 = 0$ is contained in the last integral in Eq. (8.84). To leading order the integrand is odd in $k_0$ for small $k_0$, and a cancellation thus occurs. The next leading term produces a finite result when integrated over $k_0$. For $p_z = 0$ there is a cancellation between the leading terms from the first and last integrals in Eq. (8.84), and the result is still finite.

We have plotted the different contributions to the self-energy in the high-field limit $\{eB \gg m^2,\ p_z^2,\ \mu^2,\ T^2\}$: in Fig. 4 as a function of the magnetic field, in Fig. 5 as a function of the temperature, and in Fig. 6 as a function of the momentum in the $z$-direction parallel to the magnetic field. The self-energy is even in $p_z$. In each case the self-energy is small compared to the electron rest mass and thus will only provide a small energy shift. In the

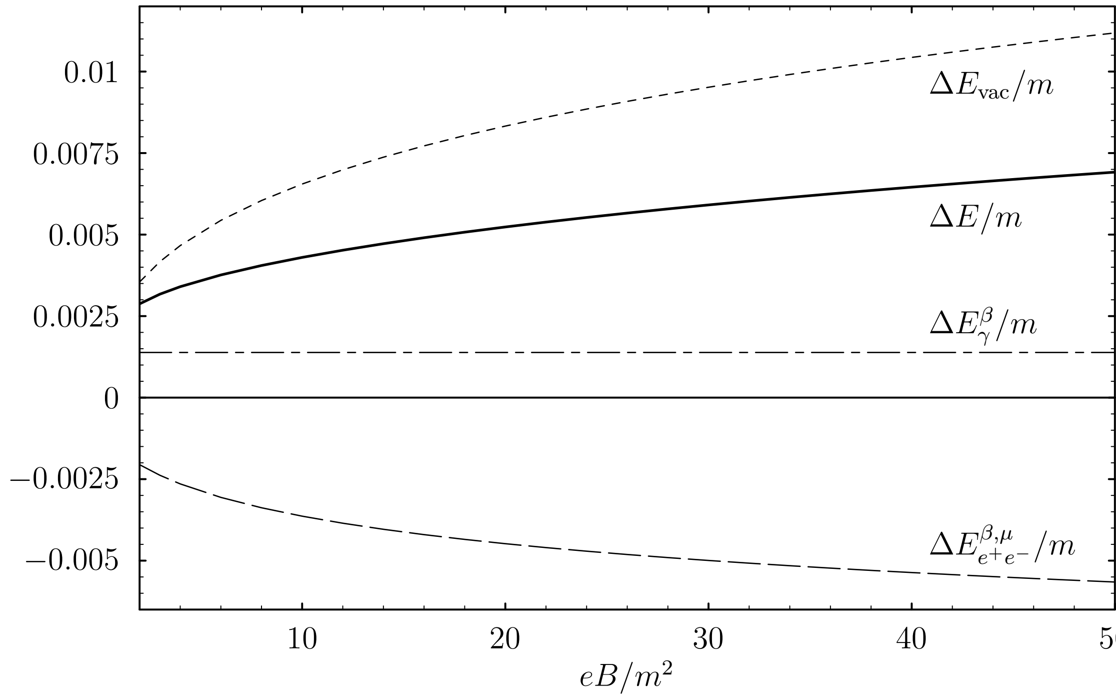


**Figure 4**: *The electron self-energy* $\Delta E = \Delta E_{\rm vac} + \Delta E_{e^+e^-}^{\beta,\mu} + \Delta E_\gamma^\beta$ *in the high-field limit* $\{eB \gg m^2,\ p_z^2,\ \mu^2,\ T^2\}$, *as a function of the magnetic field, for* $p_z/m = 1$, $\mu/m = 1$, *and* $T/m = 1$.

high-field limit it is the vacuum contribution that dominates. Using Eq. (8.79) we see that for the self-energy to be of the same order of magnitude as the electron mass we must have $eB/m^2 \simeq 10^{17}$. This corresponds to the mind-bogglingly large field of $B \simeq 10^{27}$ T.

## 8.2 The gauge-fixing dependence

We formally show in Appendix D that the self-energy is independent of the gauge-fixing parameter $\xi$ on the tree-level mass shell. In the limit of extremely strong fields we may explicitly verify this conclusion. We shall here consider the different contributions to the self-energy appearing for non-vanishing $\xi$ in Eq. (2.7). To regularize the vacuum contribution we assume the external electron to be off-shell

$$E^2 - m^2 - p_z^2 \equiv \Delta \quad , \tag{8.85}$$

and we shall find that the gauge-dependent part is proportional to $\Delta$, as shown formally in [52]. Factorizing out $\xi$, the vacuum contribution reads

$$\xi \Delta E_{\text{vac},\xi}(B \to \infty) \simeq \xi\, i \frac{e^2}{(2\pi)^3} \frac{2eB}{2m(E+p_z)} \int_{-\infty}^{\infty} dk_0\, dk_z \int_0^{\infty} du e^{-u}$$
$$\times \Big\{ (k_0 + k_z)\Delta^2 + (k_0^2 - m^2 - k_z^2 + i\varepsilon)[-2E^3 - (E^2 - m^2)k_z +$$

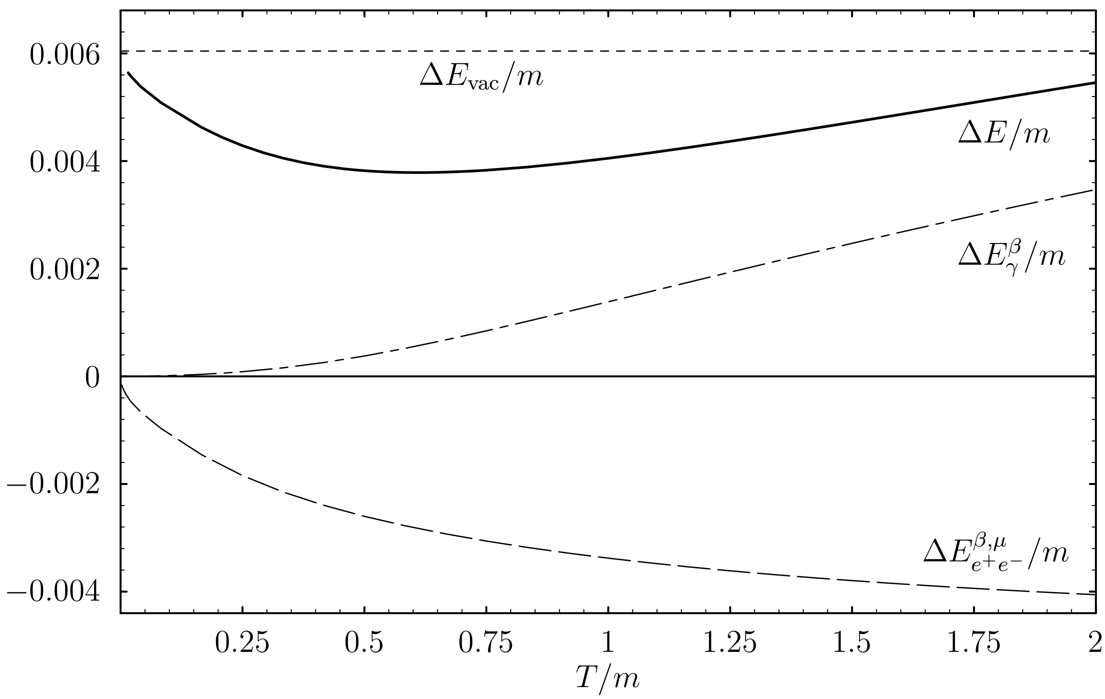


**Figure 5**: *The electron self-energy* $\Delta E = \Delta E_{\rm vac} + \Delta E_{e^+e^-}^{\beta;\mu} + \Delta E_\gamma^\beta$ *in the high-field limit* $\{eB \gg m^2,\ p_z^2,\ \mu^2,\ T^2\}$, *as a function of the temperature, for* $eB/m^2 = 8$, $p_z/m = 1$, *and* $\mu/m = 1$.

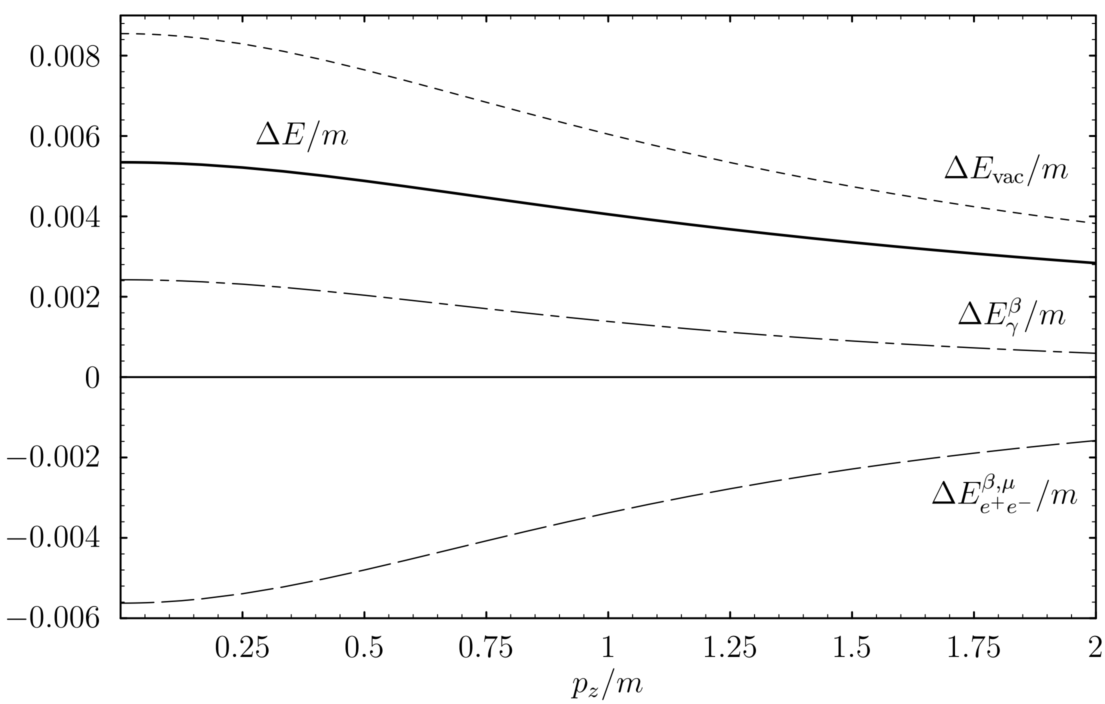


**Figure 6**: *The electron self-energy* $\Delta E = \Delta E_{\rm vac} + \Delta E_{e^+e^-}^{\beta;\mu} + \Delta E_\gamma^\beta$ *in the high-field limit* $\{eB \gg m^2,\ p_z^2,\ \mu^2,\ T^2\}$ *as a function of the momentum parallel to the magnetic field, for* $eB/m^2 = 8$, $\mu/m = 1$, *and* $T/m = 1$. *Each contribution is even in* $p_z$.

$$+(E^2+m^2)k_0 + 2p_z\{E(k_0-k_z)-E^2+p_z^2\} + p_z^2(2E+k_0-k_z)]\Big\}$$

$$\times\frac{1}{(k_0^2-m^2-k_z^2+i\varepsilon)[(k_0-E)^2-(k_z-p_z)^2-2eBu+i\varepsilon]} \quad . \tag{8.86}$$

Let us now use the Feynman parametrization

$$\frac{1}{a^2b} = 2\int_0^1 ds \frac{s}{[(1-s)b+sa]^3} \quad , \tag{8.87}$$

in the first term on the right-hand side of Eq. (8.86). We may then integrate over $k_0$ and $k_z$, to obtain

$$\Delta E_{\mathrm{vac},\xi}(B\to\infty) \simeq \frac{\Delta}{m}\frac{\alpha}{4\pi}\int_0^\infty du e^{-u}\left\{\frac{1}{u-i\varepsilon}+\right.$$

$$\left. + 2eB\Delta\int_0^1 ds \frac{s^2}{[m^2(1-s)^2-\Delta s(1-s)+2eBus-i\varepsilon]^2}\right\} \quad . \tag{8.88}$$

Notice that we cannot naively let $\Delta = 0$ here, since the result would be an ill-defined product of $\Delta$ and a logarithmic divergence. The $u$ integral is dominated by small $u$. The $s$ integral would be dominated by $s \simeq 1$. Inserting $s = 1$, except in the terms $(s-1)$, we may easily perform the $s$ integral. The result will contain simple polynomials and logarithms. Expanding the logarithms for $um^2/eB \ll 1$ to leading order, several cancellations will occur. The result reads

$$\Delta E_{\mathrm{vac},\xi}(B\to\infty) \simeq -\frac{\alpha}{4\pi}\frac{\Delta^2}{m(2m^2-\Delta)}\int_0^\infty du \frac{e^{-u}}{u+(m^2-\Delta)/2eB-i\varepsilon} \quad . \tag{8.89}$$

We may here identify an exponential integral, and use its asymptotic expansion to get

$$\Delta E_{\mathrm{vac},\xi}(B\to\infty) \simeq -\frac{\alpha}{4\pi}\frac{\Delta^2}{m(2m^2-\Delta)}\ln\left(\frac{2eB}{m^2-\Delta}\right) \quad . \tag{8.90}$$

This is obviously vanishing on the tree-level mass shell $\Delta = 0$. However, notice that if we would like to solve the dispersion relation self-consistently in this limit, the gauge-fixing dependence is only logarithmically suppressed compared to the result in the Feynman gauge, proportional to $[\ln(2eB/m^2)]^2$ according to Eq. (8.79). When considering the contribution of thermal electrons to the $\xi$ dependence, the integrand in correspondence to Eq. (8.86) is proportional to

$$\delta(k_0^2-m^2-k_z^2)\{(k_0+k_z)\Delta^2 + (k_0^2-m^2-k_z^2)[-2E_0^3+\ldots]\} \quad . \tag{8.91}$$

This is vanishing on the tree-level mass shell $\Delta = 0$. Also for thermal photons we may immediately use the on-shell energy. The contribution to the self-energy multiplied by $\xi$

then becomes

$$\Delta E^{\beta}_{\gamma,\xi} \simeq \frac{\alpha}{\pi} eB \frac{E_0}{m} \int_0^\infty du\, e^{-u} \int_{-\infty}^{\infty} dk_0\, dk_z\, k_0 \delta'(k_0^2 - k_z^2 - 2eBu) f_B(k_0) \quad , \tag{8.92}$$

where the prime on the $\delta$ function denotes a derivative with respect to its argument. Since the photon is its own antiparticle, $f_B(k_0)$ has to be even in $k_0$. Then $\Delta E^{\beta}_{\gamma,\xi}$ is vanishing due to the antisymmetric integration in $k_0$.

# 9 Discussion and final remarks

In this paper we have presented a detailed account on the fermion self-energy when both an external magnetic field and the presence of a heat bath have to be taken into account. In the context of astrophysics, many QED processes in the presence of strong magnetic fields have been studied, however mostly without the presence of a heat bath.

As was mentioned in Appendix B it has been argued in the literature that one must use a Dirac spinor basis which diagonalizes in the Sokolov–Ternov spin operator [53, 54] in order to consistently treat unstable excited relativistic Landau levels in perturbation theory [55, 56]. We have verified that such a basis can be constructed from the Dirac spinor basis defined in Appendix A. We have also seen that when a heat bath is present the situation is more complex and that in general such a basis does not diagonalize the self-energy except in the case of zero momentum parallel to the external magnetic field.

The effect of thermal quasi-particles on the emission of neutrinos of a very hot or dense star, but without the presence of a magnetic field, has been studied in [57]. It was found that under certain physical conditions the electron–plasmino annihilation may exceed the electron–positron process into neutrinos. In principle it is possible to carry out a similar analysis in the presence of a magnetic field. As we have seen in Section 5, the presence of a magnetic field leads to a $p_z$ asymmetric quasi-particle spectrum. We have suggested that this asymmetry may play a role also in dynamics, leading to the observed high space velocities of pulsars [33, 34].

We have also carried out a detailed analysis of the anomaly of the magnetic moment $\delta g/2$ when a thermal heat bath of photons, electrons and positrons is present. Previous considerations [35, 36] of this problem have led to ill-defined IR behaviour. With thermal photons, electrons and positrons present, we obtain the following contribution to the

anomaly:

$$
\begin{aligned}
\frac{\delta g^{\beta,\mu}_{e^+e^-,\gamma}}{2} \quad = \quad & -\frac{2\alpha\pi}{9}\frac{T^2}{m^2} \\
& -\frac{\alpha}{3\pi}\int_m^\infty \frac{d\omega}{\sqrt{\omega^2-m^2}} \\
& \times\left[\left(\frac{2\omega^2+2m\omega-m^2}{m^2}-\frac{m}{(\omega+m)}\right)f_F^+(\omega)+2m\frac{df_F^+(\omega)}{d\omega}\right]+ \\
& +\frac{\alpha}{3\pi}\int_m^\infty \frac{d\omega}{\sqrt{\omega^2-m^2}}\frac{2\omega^3-3m^2\omega-2m^3}{m^2(\omega+m)}f_F^-(\omega) \quad . \qquad (9.93)
\end{aligned}
$$

To the best of our knowledge this is the first IR well-defined expression for the anomalous magnetic moment when a thermal environment of photons and electrons is present. At zero temperature the $\omega$-integral in Eq. (9.93) can be carried out explicitly, and we find

$$
\frac{\delta g^{\beta,\mu}_{e^+e^-}}{2} = -\frac{\alpha}{3\pi}\left[\left(2+\frac{\mu}{m}-\frac{m}{m+\mu}\right)\sqrt{\left(\frac{\mu}{m}\right)^2-1}-\frac{2m}{\sqrt{\mu^2-m^2}}\right] \quad , \qquad (9.94)
$$

for $\mu > m$. In the limit of large $\mu$, this agrees with the result in [35], as could be expected since the IR sensitivity is sub-dominant. Thus the conclusion in [35] that $g$ can become considerably smaller than 2 for high densities remains valid. Using a bold extrapolation to large corrections we find that $g+\delta g \simeq 0$ for $\mu \simeq 35\, m_e$, which should be compared with a typical chemical potential $\mu \simeq 300\, m_e$ inside a neutron star. Even if the approximations are not valid for such a large $\mu$ there could still be important corrections to the synchrotron radiation from the surface of a neutron star, and thus to the estimation of the $B$-field from observations.

We notice also in Eq. (9.94) that $\delta g$ seems to become very large for small densities, e.g. when $\mu \to m$. This is an artefact of the expansion in $B$. As discussed in Section 4 we do not expect an expansion in $B$ to be universally possible, but some sort of de Haas–van Alphen oscillations should show up in certain limits. For $T=0$ and $B$ fixed we can take the limit $\mu \to m$ directly in Eq. (2.14) and it corresponds actually to a strong-field calculation since only the lowest Landau level is inside the Fermi sea. The leading contribution can then be extracted from Eq. (8.81) using Eq. (E.31) and we obtain

$$
\Delta E = -\frac{\alpha}{\pi}\sqrt{\mu^2-m^2}\left[\ln\left(\frac{2eB}{\mu^2-m^2}\right)+2-C\right]+\mathcal{O}\left(\frac{\mu^2-m^2}{eB}\right) \quad , \qquad (9.95)
$$

valid for $m < \mu < \sqrt{m^2+eB}$. Our conclusion is that the energy shift is perfectly finite when $\mu \to m$ (in fact, it goes to zero), but since $\Delta E$ does not admit a power series

expansion in $B$ the standard definition of a magnetic moment (see Eq. (7.62)) does not make sense.

In a strong magnetic field we have, finally, observed the presence of an imaginary part in the self-energy which becomes field-independent if the external magnetic field is large enough. Formally, this imaginary part appears in a way similar to the pole description of Landau in the theory of longitudinal plasma oscillations [51]. This imaginary contribution is therefore a relativistic counterpart of Landau damping and is physically due to the possibility of $e^+e^-$ annihilation into a *single* photon in a magnetic field [50].

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# APPENDICES

## A External-field propagator

In this appendix we summarize, for the convenience of the reader, the relevant expressions used in the main text for a constant magnetic field $B$ in the negative $z-$direction in the gauge $A_\mu = (0, 0, Bx, 0)$. We use the $\gamma$-matrices in the chiral representation,

$$\gamma_0 = \begin{pmatrix} 0 & -1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad . \tag{A.1}$$

We seek solutions to the Dirac equation for a fermion in an external field

$$(i \not{D} - m)\Psi_\kappa^{(\pm)}(\mathbf{x}, t) = 0 \quad , \tag{A.2}$$

where $iD\!\!\!/ = \gamma^\mu(i\partial_\mu + eA_\mu)$ for a particle of charge $-e$. With our choice of gauge it follows that

$$\Psi^{(\pm)}_\kappa(\mathbf{x},t) = \frac{1}{2\pi\sqrt{2E}}\,\exp[\pm i(-Et+p_y y+p_z z)]\,\Phi^{(\pm)}_\kappa(x) \quad , \tag{A.3}$$

where $\kappa$ denotes $p_y$, $p_z$, and any other quantum-number necessary to specify the wave functions. For ease of notation we shall write $iD_\mu \to \Pi_\mu = (E, -p_z, -\mathbf{\Pi}_\perp)$ when acting on the above wave functions. Acting with $(\Pi\!\!\!/ + m)$ on Eq. (A.2), using $[\Pi_x, \Pi_y] = ieF_{xy} = ieB$, we find

$$(E^2 - m^2 - p_z^2 - \Pi_\perp^2 + eB\sigma_{xy})\Psi = 0 \quad , \tag{A.4}$$

where $\sigma_{xy} \equiv i[\gamma_x, \gamma_y]/2 = \mathrm{diag}[\sigma_z, \sigma_z]$. Let us now introduce the functions

$$I_{n;p_y}(x) \;\equiv\; \left(\frac{eB}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}eB\left(x-\frac{p_y}{eB}\right)^2\right]\frac{1}{\sqrt{n!}}H_n\left[\sqrt{2eB}\left(x-\frac{p_y}{eB}\right)\right] \quad , \tag{A.5}$$

where $H_n$ is the Hermite polynomial given by the Rodrigues formula as

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2}\frac{d^n}{dx^n}e^{-\frac{1}{2}x^2} \quad , \tag{A.6}$$

and we define $I_{-1;p_y}(x) = 0$. The functions $I_{n;p_y}(x)$ are normalized according to

$$\int dx I_{n;p_y}(x) I_{n';p_y}(x) = \delta_{n,n'} \quad , \tag{A.7}$$

for $n, n' = 0, 1, 2, \ldots$ . Defining $\xi_\pm = \Pi_x \mp i\Pi_y$, we have $\Pi_\perp^2 = \xi_+\xi_- + eB$, and they act on the $I_n$ functions according to

$$\begin{aligned} \xi_+ I_{n-1;p_y}(x) &= i\sqrt{2eBn}I_{n;p_y}(x) \ , \\ \xi_- I_{n;p_y}(x) &= -i\sqrt{2eBn}I_{n-1;p_y}(x) \ . \end{aligned} \tag{A.8}$$

It readily follows that

$$(\Pi_\perp^2 - eB\sigma_z)\,\mathrm{diag}[I_{n;p_y}(x), I_{n-1;p_y}(x)] = 2eBn\,\mathrm{diag}[I_{n;p_y}(x), I_{n-1;p_y}(x)] \quad . \tag{A.9}$$

We may therefore write $\Phi$ in Eq. (A.3) in the form

$$\Phi^{(\pm)}_{\zeta;n,p_y,p_z}(x) = \mathrm{diag}[I_{n;p_y}(x), I_{n-1;p_y}(x), I_{n;p_y}(x), I_{n-1;p_y}(x)]u^{(\pm)}_{\zeta,n,p_y,p_z} \quad , \tag{A.10}$$

where $u^{(\pm)}_\kappa$ is a Dirac spinor independent of $x_\mu$, and $\zeta = \pm 1$ denotes a polarization index. The energy eigenvalues are given by the relativistic Landau levels

$$E = E_n(p_z) = \sqrt{m^2 + p_z^2 + 2eBn} \quad . \tag{A.11}$$

In the lowest Landau level ($n = 0$), $I_{-1;p_y} \equiv 0$ implies that there is only one choice of $u_\kappa$ possible, corresponding to $\zeta = 1$. For the higher Landau levels there is a twofold degeneracy, since two linearly independent $u^{(\pm)}_{\zeta,n,p_y,p_z}$ may be found, and there is thus an ambiguity in the choice of wave functions.

In [58] there is a discussion about a choice of the $u^{(\pm)}_{\zeta,n,p_y,p_z}$ which diagonalizes the self-energy corrections in vacuum. We have not found any simple generalization of that basis at finite temperature, as stated in Appendix B. On the other hand, in this paper we need no such basis since we always deal either with the expectation value of the self-energy in the non-degenerate lowest Landau level or with the full $4 \times 4$ matrix. For any higher Landau levels it is, of course, mandatory to diagonalize the matrix including corrections since the perturbation theory is degenerate, as emphasized e.g. in [14].

We shall here use the choice of wave functions given in [12]

$$\Phi^{(+)}_{1;n,p_y,p_z}(x) = \frac{1}{\sqrt{E_n + p_z}} \begin{pmatrix} (E_n + p_z) I_{n;p_y}(x) \\ -i\sqrt{2eBn}\, I_{n-1;p_y}(x) \\ -m I_{n;p_y}(x) \\ 0 \end{pmatrix} \quad . \tag{A.12}$$

$$\Phi^{(+)}_{-1;n,p_y,p_z}(x) = \frac{1}{\sqrt{E_n + p_z}} \begin{pmatrix} 0 \\ -m I_{n-1;p_y}(x) \\ -i\sqrt{2eBn}\, I_{n;p_y}(x) \\ (E_n + p_z) I_{n-1;p_y}(x) \end{pmatrix} \quad , \tag{A.13}$$

$$\Phi^{(-)}_{1;n,p_y,p_z}(x) = \frac{1}{\sqrt{E_n - p_z}} \begin{pmatrix} -m I_{n;-p_y}(x) \\ 0 \\ (-E_n + p_z) I_{n;-p_y}(x) \\ i\sqrt{2eBn}\, I_{n-1;-p_y}(x) \end{pmatrix} \quad , \tag{A.14}$$

$$\Phi^{(-)}_{-1;n,p_y,p_z}(x) = \frac{1}{\sqrt{E_n - p_z}} \begin{pmatrix} i\sqrt{2eBn}\, I_{n;-p_y}(x) \\ (-E_n + p_z) I_{n-1;-p_y}(x) \\ 0 \\ -m I_{n-1;-p_y}(x) \end{pmatrix} \quad . \tag{A.15}$$

It can be shown that the collection of all $\Psi$'s forms a complete orthonormal set. The wave functions are normalized according to

$$\int d^3x \Psi^{(\lambda)\dagger}_{\zeta;n,p_y,p_z}(x) \Psi^{(\lambda')}_{\zeta';n',p'_y,p'_z}(x) = \delta_{\zeta,\zeta'}\, \delta_{n,n'}\, \delta_{\lambda,\lambda'}\, \delta(p_y - p'_y)\, \delta(p_z - p'_z) \quad , \tag{A.16}$$

where $\lambda = \pm$. The propagator in the external field $iS(x', x) \equiv iS_{\mathrm{vac(x',x)}} + iS^{\beta,\mu}(x', x)$, as given in Eqs. (2.3), (2.4), is then explicitly written (see e.g. [12] for the vacuum part,

and [4] for the additional term when we have fermions according to some one-particle distribution):

$$
\begin{aligned}
S(x',x)_{ab} \;=\; & \sum_{n=0}^{\infty}\int \frac{dk_0\,dk_y\,dk_z}{(2\pi)^3}\exp[-ik_0(t'-t)+ik_y(y'-y)+ik_z(z'-z)] \\
& \times\left[\frac{1}{k_0^2-k_z^2-m^2-2eBn+i\varepsilon}+2\pi i\delta(k_0^2-k_z^2-m^2-2eBn)f_F(k_0)\right] \\
& \times S_{ab}(n;k_0,k_y,k_z;x',x) \quad , \qquad \text{(A.17)}
\end{aligned}
$$

where

$$
f_F(k_0)=\Theta(k_0)f_F^+(k_0)+\Theta(-k_0)f_F^-(-k_0) \quad , \qquad \text{(A.18)}
$$

The matrix $S(n;k_0,k_y,k_z,x',x)$ entering above is in the chiral representation of $\gamma$-matrices explicitly written

$$
S(n;k_0,k_y,k_z)\equiv
$$

$$
\begin{pmatrix}
mI_{n,n} & 0 & -(k_0+k_z)I_{n,n} & -i\sqrt{2eBn}I_{n,n-1} \\
0 & mI_{n-1,n-1} & i\sqrt{2eBn}I_{n-1,n} & -(k_0-k_z)I_{n-1,n-1} \\
-(k_0-k_z)I_{n,n} & i\sqrt{2eBn}I_{n,n-1} & mI_{n,n} & 0 \\
-i\sqrt{2eBn}I_{n-1,n} & -(k_0+k_z)I_{n-1,n-1} & 0 & mI_{n-1,n-1}
\end{pmatrix} , \qquad \text{(A.19)}
$$

where we have used the short-hand notation

$$
I_{n',n}\equiv I_{n';k_y}(x')I_{n;k_y}(x) \quad . \qquad \text{(A.20)}
$$

# B The Sokolov–Ternov spin operator

Let $H_D=\boldsymbol{\alpha}\cdot\mathbf{\Pi}+\beta m$, where $\alpha^i=\gamma^0\gamma^i,\ \beta=\gamma^0$, be the Landau–Dirac Hamiltonian in an external magnetic field. Let the magnetic field be parallel to the $z$-axis. The $z$-component of the Sokolov–Ternov [53] spin operator is $\hat{\mu}_z=\sigma_z(m+\not{\Pi}_\perp)$. (For a recent discussion of this operator in the context of calculations of the electron anomalous magnetic moment, see [54].) It follows that $\hat{\mu}_z$ is conserved, i.e. it commutes with $H_D$. It is easy to construct linear combinations of the solutions in Eq. (A.3), which also diagonalize $\hat{\mu}_z$. Suppressing all the quantum numbers of the solutions in Eq. (A.3) except the polarization $\zeta=\pm1$, solutions to the Dirac equation (A.2) that diagonalize $\hat{\mu}_z$ can be written in the form

$$
\Psi^{(\pm)}_{\hat{\mu}_z}=N_\pm(\Psi_+-ia_\pm\Psi_-) \; , \qquad \text{(B.21)}
$$

where, if $n \neq 0$,

$$a_\pm = \frac{p_n}{m \pm E_n(0)} \; ; \quad N_\pm = \frac{p_n^2}{p_n^2 + (m \pm E_n(0))^2} \; , \tag{B.22}$$

where $p_n = \sqrt{2eBn}$. The states in Eq. (B.21) have the property $\hat{\mu}_z \Psi^{(\pm)}_{\hat{\mu}_z} = \pm\sqrt{m^2 + p_n^2}\Psi^{(\pm)}_{\hat{\mu}_z}$. The lowest Landau level is diagonal in $\hat{\mu}_z$ with eigenvalue $m$. It can be verified that the propagator $S(x', x)$, when calculated in the basis in Eq. (B.21), still is the same as the one given in Appendix A. The basis in Eq. (B.21) furthermore has the property that the self-energy $\Sigma(x', x)$ is diagonal in the vacuum sector to $\mathcal{O}(\alpha)$. By using the fact that the on-shell Sokolov–Ternov operator can be written in the form $\hat{\mu}_z = \sigma_z(\gamma^0 p^0 - \gamma_z p_z)$, this follows easily from the fact that $[\hat{\mu}_z, \beta\hat{\Sigma}_0] = 0$, where $\hat{\Sigma}_0$ is the vacuum self-energy operator [9]. In the presence of a thermal heat bath the self-energy operator $\beta\hat{\Sigma}^{\beta,\mu}$, defined as the sum of Eq. (3.32) and Eq. (3.34), commutes with $\hat{\mu}_z$ only if $p_z = 0$. It has been observed [55, 56] that in the vacuum sector one must use the basis where $\hat{\mu}_z$ is diagonal in order to introduce consistently in perturbation theory, a resonance width into the QED cyclotron scattering amplitudes. In a thermal environment the situation is more complex and further studies are required.

# C The tadpole

Also the tadpole could possibly contribute to the one-loop self-energy. In configuration space the tadpole is proportional to

$$J^\nu(x) \equiv \mathrm{tr}\,[e\gamma^\nu iS(x,x)] \quad , \tag{C.23}$$

where the trace is over spinor indices. Introducing an $\varepsilon > 0$ in the argument of $\overline{\Psi}\; : t' \to t + \varepsilon$, we may remove the time-ordering. Then we let $\varepsilon$ vanish and immediately find

$$J^\nu(x) = -e\langle\overline{\Psi}(x)\gamma^\nu\Psi(x)\rangle \quad , \tag{C.24}$$

i.e. the expectation value of the electromagnetic current. We could instead choose to change the argument of $\Psi : \; t \to t+\varepsilon$, in order to remove the time ordering, and then let $\varepsilon$ vanish. In this case we need to use the equal time anticommutation relation $\{\Psi_a(\mathbf{x}', t), \Psi_b^\dagger(\mathbf{x}, t)\}_+ = \delta_{ab}\delta^3(\mathbf{x}' - \mathbf{x})$ to reverse the order of $\overline{\Psi}$ and $\Psi$. Since $\gamma^\nu$ is traceless, we again arrive at Eq. (C.24). Let us now separate the current in its vacuum and thermal contributions $J^\nu = J^\nu_{\rm vac} + J^\nu_{\beta,\mu}$. Trivially $J^\nu_{\rm vac}$ vanishes after renormalization, and so does $J^j_{\beta,\mu}$ , for

$j = 1,2,3$ in our static model. What is left is the charge density $J^0_{\beta,\mu}$. But in order for our static model to be valid there must be no such charge asymmetry on average. If there is a finite chemical potential for electrons there must be a compensating (static) background charge to make the average charge density vanish everywhere. Actually, it is easily shown with explicit calculations for the particular system we consider that the only possible non-vanishing part of $J^\nu$, as defined in Eq. (C.23), is $J^0_{\beta,\mu}$. Using the electron propagator in Eq. (A.17) and its thermal generalization, it follows that $J^x$ and $J^z$ vanish when performing the trace. Also $J^y$ and $J^0_{\text{vac}}$ vanish due to the antisymmetric integration in $k_y$ and $k_0$, respectively. In the case of vanishing chemical potential, $f_F(k_0)$ is even, and then also $J^0_{\beta,\mu}$ vanishes due to the antisymmetric integration. In the case of general $\mu$ we find

$$J^0_{\beta,\mu} = -e\,\frac{eB}{(2\pi)^2}\sum_{n=0}^{\infty}\int dk_z\{f_F^+[E_n(k_z)] - f_F^-[E_n(k_z)]\}(2-\delta_{n,0}) \quad , \tag{C.25}$$

i.e. the charge density. Notice the factor $(2-\delta_{n,0})$, which originates in a term $I_{n,n}+I_{n-1,n-1}$, and shows the twofold degeneracy in all but the lowest Landau levels.

# D Gauge independence

We have used the Feynman gauge throughout the article and now we shall show that the result is gauge-independent on-shell, i.e. $\langle\Psi_\kappa|\hat{\Sigma}|\Psi_{\kappa'}\rangle$ is independent of the gauge-fixing parameter $\xi$ when $\Psi_\kappa$ and $\Psi_{\kappa'}$ are solutions to the Dirac equation. The only possible $\xi$ dependence in a general covariant gauge comes from the photon propagator. In vacuum, it would give a term of the form

$$\xi\int d^4x'\,d^4x\overline{\Psi}_\kappa(x')ie^2\gamma^\mu[\partial_\mu\partial'_\nu\tilde{D}(x'-x)]S(x,x')\gamma^\nu\Psi_{\kappa'}(x)\ , \tag{D.26}$$

where

$$\tilde{D}(x'-x) = \int[d^4k]\frac{e^{-ik(x'-x)}}{(k^2+i\epsilon)^2}\ . \tag{D.27}$$

After one partial integration in $x$ or $x'$ the Dirac equation can be used together with $(i\not{D}-m)S(x,x') = \delta(x-x')$ to show the Eq. (D.26) is zero. At finite temperature the gauge-dependent part of the photon propagator is again given by a total derivative [24] and the same proof goes through.

# E Exponential integrals

In [59] we have the following definitions of the exponential integrals

$$\mathrm{E}_1(z) \equiv \int_z^\infty dt \frac{e^{-t}}{t} \quad , \ |\arg z| < \pi \quad , \tag{E.28}$$

$$\mathrm{Ei}(x) \equiv -\mathcal{P}\int_{-x}^\infty dt\, \frac{e^{-t}}{t} \quad , \ x \in \mathbb{R} \quad . \tag{E.29}$$

Furthermore they satisfy the interrelation

$$\mathrm{E}_1(-x \pm i\varepsilon) = -\mathrm{Ei}(x) \mp i\pi \quad , \ x > 0 \quad , \tag{E.30}$$

and have the series expansions

$$\mathrm{E}_1(z) = -C - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n n!} \quad , \ |\arg z| < \pi \quad , \tag{E.31}$$

$$\mathrm{Ei}(x) = C + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n n!} \quad , \ x \in \mathbb{R} \quad , \tag{E.32}$$

where $C = 0.5772156649\ldots$ is Euler's constant.

# F Electron-positron annihilation

We shall here consider the process $e^- e^+ \mapsto \gamma$, where the positron comes from the heat and charge bath. A large magnetic field may absorb momentum to make the process energetically possible. Consider an electron in the state described by $\Psi_p^{(+)}$, a positron in the state described by $\Psi_\kappa^{(-)}$, and a photon with momentum $q_\mu$ and polarization $\varepsilon_\mu(\lambda, q)$. To the lowest order in perturbation theory (i.e. perturbatively in the quantum electromagnetic field, the static uniform magnetic field is treated exactly), we find the corresponding transition matrix element

$$iT_{p,\kappa;q,\lambda} = -ie \int d^4x \overline{\Psi}_\kappa^{(-)}(x)\, \gamma^\mu \Psi_p^{(+)}(x) \varepsilon_\mu^*(q,\lambda) e^{iq\cdot x} \quad . \tag{F.33}$$

Let us now form $|T_{p,\kappa;q,\lambda}|^2$. Sum over all polarizations of the outgoing photon

$$\sum_\lambda \frac{\varepsilon_\mu^*(q,\lambda)\, \varepsilon_\nu(q,\lambda)}{\varepsilon^*(q,\lambda)\cdot\varepsilon(q,\lambda)} = g^{\mu\nu} \quad , \tag{F.34}$$

and integrate over the photon momentum

$$\int \frac{[d^3\mathbf{q}]}{2q_0}\bigg|_{q_0=|\mathbf{q}|} = \int [d^4q]\, 2\pi\, \delta(q_0^2 - \mathbf{q}^2)\, \Theta(q_0) \quad . \tag{F.35}$$

Sum over all incoming positrons with distribution $f_F^-(E_\kappa)$. Due to the normalization of our wave functions in Eq. (A.16) we must divide by $\int dy\,dz$, for the norm of the decaying electron state. Dividing by the infinite time elapsed between the final and initial state $\int dt$, we find the decay rate

$$\begin{aligned}\Gamma &= \frac{e^2}{\int dy\,dz\,dt}\int d^4x \int d^4x'\,\overline{\Psi}^{(+)}_{\zeta;n,p_y,p_z}(x)\,\gamma^\mu \sum_\kappa f_F^-(E_\kappa)\Psi_\kappa^{(-)}(x')\overline{\Psi}_\kappa^{(-)}(x)\gamma_\mu \\ &\quad\times \int [d^4q]\,2\pi\,\delta(q^2)\Theta(q_0)e^{-iq\cdot(x'-x)}\,\Psi^{(+)}_{\zeta;n,p_y,p_z}(x') \quad . \end{aligned} \tag{F.36}$$

By comparing with Eq. (2.4) we see that $\sum_\kappa f_F^-(E_\kappa)\Psi_\kappa^{(-)}(x')\overline{\Psi}_\kappa^{(-)}(x)$ is exactly the thermal positron contribution to the Dirac fermion propagator. Due to energy conservation (as follows when performing the integration over $t$) $q_0 < 0$ will not contribute, so we may drop $\Theta(q_0)$. The decay rate is thus exactly equal to the imaginary part of the electron self-energy, with thermal positrons in the intermediate states, obtained through $1/(q^2+i\varepsilon) = \mathcal{P}(1/q^2) - i\pi\delta(q^2)$ in the photon propagator. This splitting into principal and imaginary part is equivalent to what was done in the high-field limit in Section 8.1, using the definition and interrelation of exponential integrals as given in Appendix E.

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